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HIGHER ORDER EFFECTS IN TWO-DIMENSIONAL TURBULENT JETS

By

VINAY PRAKASH MATHUR

DEPARTMENT OF AERONAUTICAL ENGINEERING
INDIAN INSTITUTE OF TECHNOLOGY, KANPUR

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VINAY PRAKASH MATHUR

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CERTIFICATE

This is to certify that the work 'Higher Order Effects in Two-Dimensional Turbulent Jets' has been carried out under my supervision and has not been submitted elsewhere for a degree.

Vijay Gupta

October - 1976

(Dr. Vijay Gupta)
Department of Aero. Engg.
Indian Institute of Technology, Kanpur
(India)

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October - 1976

(Vinay Prakash Mathur)
Department of Aero. Engg.
Indian Institute of Technology,
Kanpur (India)

ABSTRACT

In this thesis an attempt is made to determine the viscous effects and contribution of pressure gradient terms to velocity profiles in two-dimensional, turbulent, submerged jet using the perturbation technique. The problem of two-dimensional jet is shown to be one of regular perturbation. The reciprocal of the Reynolds number in the vorticity transport equation for the two-dimensional turbulent flow is used as the perturbation parameter. Prandtl's eddy viscosity model which takes the eddy viscosity to be constant across the jet-width is used as the closure hypothesis. The stream function, Reynolds stress and viscous stress terms are expanded in asymptotic series. We obtain the first order and second order equations by comparing the orders of the various terms in the vorticity transport equation. The first order equation is the boundary-layer equation for zero pressure gradient and was solved by Goertler (8) for the case of two-dimensional jet. The second order equation which includes viscous effects is solved numerically. The pressure gradient term is found to be of a still higher order and thus does not affect the velocity distribution to second order.

Due to viscosity the velocity near the jet axis reduces and that away from the axis increases resulting in a flattening of the velocity profile and a increase in the width of the jet boundary layer.

There is also a slight dependence on the axial distance, the viscous corrections reducing as we move away from the slit. These results are valid and significant in the Reynolds number range $10^2 < Re < 10^4$ and for x-stations $8 < x/d < 25$. For $x/d > 25$ and $Re > 10^4$ the first order velocity profile (based on infinitely large Reynolds number and full development assumptions) can be taken as valid. For $x/d < 8$ and $Re < 10^2$, the jet model breaks down and the results obtained are no longer valid.

TABLE OF CONTENTS

List of Symbols

A	–	Constant scaling factor for velocity.
B	–	Constant scaling factor for length.
b	–	Jet half-width.
C	–	Constant of proportionality for eddy viscosity coefficient.
d	–	Width of jet slit.
E_1, E_2	–	Sequence of gauge functions for stream function.
G	–	Gauge function for eddy viscosity coefficient.
J	–	Momentum flux in the axial direction.
K	–	Eddy viscosity coefficient.
ℓ	–	Mixing length.
R	–	Reynolds number.
u	–	Mean axial velocity component.
u'	–	Fluctuating axial velocity component.
u_o	–	Mean centre line velocity.
U	–	Non-dimensional mean axial velocity component.
U_o	–	Non-dimensional mean centre line velocity.
U_e	–	Non-dimensional jet exhaust velocity.
v	–	Mean transverse velocity component.
v'	–	Fluctuating transverse velocity component.
V	–	Non-dimensional mean transverse velocity component.
x'	–	Axial coordinate.
x	–	Non-dimensional axial coordinate.
X	–	Scaled non-dimensional axial coordinate.

y'	–	Transverse coordinate.
y	–	Non-dimensional transverse coordinate.
y_c	–	Transverse distance from axis at which the axial velocity is half the centre line velocity.
Y	–	Scaled non-dimensional transverse coordinate.
δ	–	Boundary layer thickness.
Δ	–	A small parameter, function of Reynolds number.
ψ	–	Stream function.
ρ	–	Density.
η	–	A non-dimensional similarity parameter.
ε	–	A small parameter.
τ	–	Reynolds stress components.

CHAPTER 1

INTRODUCTION

INTRODUCTION

Theoretical prediction of mechanism of transfer of mass, momentum etc. and velocity profiles in turbulent jet flows, as accurately as possible, is important for optimization of jet engine performance while minimizing noise generation. It is also necessary for a better understanding of mechanism of turbulent diffusion in general which would be useful in designing combustion chambers, jet pumps, mixing tanks and disposal of waste gases.

The majority of analytical and experimental research conducted on turbulent jets has been concentrated on circular jets. This is mainly because of the fact that circular jets are of practical interest in engine exhaust configurations. Never the less the two dimensional geometry also offers many opportunities for reaching a better understanding of the problem of free-turbulent flows, that is turbulent flows in absence of solid boundaries. Though plane turbulent wakes have been extensively studied by Townsend(23), Grant (9) etc. this is not so with plane jets.

The need to improve our understanding of the two-dimensional turbulent jet has also arisen with the development of short take-off and landing aircrafts for which aerodynamic designers are considering the use of various blown-flap concepts to provide the lift augmentation required for these types of aircrafts. A slot jet above the flaps may be used in such aircrafts.

In the analytical formulation of free turbulent shear flows the viscous effects have so far been totally neglected (20). This gave reasonable agreement with experimental results for high Reynolds-number flows (8). This may not be so when lower Reynolds-number flows are considered. It is also worthwhile to estimate the order of error introduced because of neglecting pressure-gradient terms in free jets. The development of perturbation techniques as applicable to fluid mechanics permit an attempt to answer the above questions. Similar studies have in the past been conducted by researchers like K.S. Yajnik (24) and Noor Afzal (12) on wall turbulent flows (boundary layer flows, channel flow, pipe flow etc.) But free shear flows had somehow been left out. In this work we have applied the perturbation procedure to one case of free-shear flows, namely, two-dimensional turbulent jet.

One of the pioneers in the field of jet flows is H. Schlichting who starting from the fundamental viscous, incompressible flow equations obtained by numerical technique the velocity distribution across laminar jets, both plane and round. Analytical solutions were obtained later for this case by Bickley (3) and Goldstein (7). The velocity distribution across turbulent jets has been investigated by various workers. In analytic formulation of turbulent shear flows, the equations of motion contain too many unknows to be solved completely. To obtain the results one introduces some sort of a closure hypothesis about the nature of turbulent stresses. Most of

the successful hypothesis define and use the concept of a mixing length. The concept of mixing length was first introduced by Prandtl in 1925. He defined it as that distance in a direction transverse to the flow which must be covered by a lump of fluid particles travelling with its original mean velocity in order to make the difference between this velocity and the velocity at the new location equal to the transverse fluctuations in velocity. If the momentum is preserved over this length, then the arrival of this particle at the new location will appear as a fluctuation in the velocity at that point. The concept of mixing length is some what analogous with that of mean free path in kinetic theory of gases. Other researchers have used variations of the same basic idea.

Some of the more successful closure hypothesis used for stresses in free turbulent shear flows are given below :

i) Prandtl's old theory of momentum transfer (13)

ii) Taylor's vorticity transfer theory (10)

iii) Prandtls new theory of momentum transfer (15)

iv) Reichardt's molecular analogy theory. (17)

Prandtl's old theory assumes that the momentum mixing length is constant in the transverse direction, but increases linearly in the longitudinal direction. The first assumption uses the fact that there are no walls in the flow field. The mixing length decreases near the walls (at walls it must be zero). Tollmein (22) applied Prandtl's old theory to solve the following three problems

a) The boundary layer of an infinite plane parallel jet

b) Parallel jet issuing from a narrow orifice

c) Axially symmetric jet issuing from a small orifice.

He was able to obtain quite satisfactory velocity fields. One necessary conclusion of this hypothesis is the similiarity of the temperature and the velocity fields; this is grossly erroneous.

Taylor sought to correct the above by assuming that the Reynolds stress is determined not by transfer of momentum but by transfer of vorticity. He defined a vorticity mixing length as the distance over which vorticity of fluid particles remains constant during transfer from one location to another. Howarth (11) applied Taylors assumption to determine the velocity distribution across jets. The results were identical with those of Tollmeins. However the temperature profiles obtained by Taylor's theory were confirmed by experiments.

With both these theories the velocities near the centre are substantially in error. To over come this difficulty Prandtl later argued that it is not the mixing length which remains constant across the jet but the coefficient of turbulent momentum exchange remains constant. Further he proposed that this coefficient of turbulent viscosity or momentum exchange is proportional to the density, the maximum velocity difference and the jet thickness.

Goertler (8) solved the problem of plane submerged jet using Prandtl's new theory and obtained for the velocity distribution across the jet

$$\frac{U}{U_o} = (1 - \tanh^2 \eta) \qquad \text{where}$$

U = Local velocity

U_o = Velocity on jet centre line

η = Similarity parameter $\sigma Y/X$

Y = Lateral distance of the point from jet axis

X = Distance of cross section from the jet origin

σ = A constant determined experimentally.

He neglected the viscous stress completely and assumed that the pressure is constant everywhere.

Reichardt assumed that diffusion process for masses of fluid in turbulent motion is analogous to that in molecular motion. He postulated that the lateral transport of momentum is proportional to the transverse gradient of the horizontal component of momentum and obtained similarity in the mean value of dynamic pressure rather than in velocity profiles. Velocity distribution obtained by him is given by

$$\frac{U^2}{U_o^2} = e^{-\text{const}\ \eta^2} \qquad \text{where}$$

$\eta = Y/X$.

Some of the earlier experimental investigations on plane jets were carried out by Bicknell (4), Forthmann (6) and Reichardt (16) and later by L.J.S. Bradsbury (2).

Forthmann used a rectangular slit of 3 cm × 63 cm and a discharge velocity U_e of 3500 cm/sec. The velocity at any point was determined

from a total head tube (not corrected for error due to turbulence). Forthmann only measured the velocity in the plane of symmetry perpendicular to the slit.

Reichardt used a rectangular slit of 0.7 cm × 15 cm and a discharge velocity of 5000 cm/sec. In these experiments the Reynolds numbers specified as $Re = U_e d/\nu$ amounted to 70000 and 23000respectively. From Forthmann's experiment the mixing length from the momentum transfer theory is

$$\ell = c\, X = 0.0165\, X$$

Bradsbury used a slot of 3/8" × 18" and the jet exhausted into a slow moving stream of air. He compared results with those of Townsends wake investigations. It was found that self similiarity in velocity profiles is established about 30 jet widths downstream of the nozzle.

All the investigations reported in the literature concern themselves with studying the velocity profiles for high Reynolds stress (viscous term negligible) and for large distances away from the orifice (so that complete self similarity is assumed).

This present investigation attempts to include the effect of viscous stresses as perturbations to the previously obtained solutions valid for very high Reynolds numbers. The results obtained are applicable to moderate Reynolds numbers. A weak dependence on x is also obtained by the perturbation procedure.

CHAPTER 2

INTRODUCTION TO PERTURBATION TECHNIQUES

INTRODUCTION TO PERTURBATION TECHNIQUES

Introduction

Even after a mathematical model of a physical problem has been constructed, obtaining exact solutions is a difficult task in most engineering situations. The equations encountered may be non-linear, coupled, of very high order or may have singularities which make obtaining analytic or even numerical solutions very difficult. In such situations one often resorts to some or the other of approximation method. One powerful approximation method consists of the expansion of solution in a power series in a parameter. This is the classical perturbation technique upon which much of the edifice of science rests.

The basic premise on which perturbation technique depends is the hypothesis that small causes produce small effects. Thus, if viscous forces are small compared to other forces in the problem, the solution obtained on neglecting the viscosity should differ only a little from the exact solution. The result can then be modified by including the viscous forces estimated by the derivative of the approximate velocities so obtained. The correction thus obtained is termed the second order term in the expansion of results in terms of the viscosity parameter. An iterative or a perturbation scheme is thus set up. The first approximation is thus valid for very low viscosity effects (very high Reynolds numbers), and

inclusion of higher order terms extends the validity into lower Reynolds numbers.

Of course the basic premise of such a perturbation scheme has limitations. One such limitation applies when the neglecting of one of the terms changes the character of the governing equations drastically. Such a condition is called singular behaviour and examples of this include the boundary layers on blunt bodies which give rise to D' Alembert's paradox.

To be able to apply perturbation methods one must be able to compare the orders of terms. It is helpful to introduce a formal concept of ordering.

Ordering :

Consider a class of function $f(\varepsilon)$ of a small parameter ε, such that

i) f is real and continuous

ii) For each f there exists an interval for which the function is positive (of the type $0 \leq \varepsilon \leq \varepsilon_0$).

iii) We take $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$ that is limit of the function as ε approaches zero is zero.

The order of these functions refers to the relative rate at which these approach zero. To be able to express these rates qualitatively, one compares the limiting behaviour of the functions f with a set of gauge functions. These gauge functions $g(\varepsilon)$ consist

of an ordered set of functions of ε which are so familiar that their limiting behaviour can be regarded as known.

If we take limit of f/g as ε tends to zero there are three possibilities

$$\text{i)} \quad \lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{g(\varepsilon)} \to 0 \tag{2.1}$$

$$\text{ii)} \quad \lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{g(\varepsilon)} \to \text{Constant} \tag{2.2}$$

$$\text{iii)} \quad \lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{g(\varepsilon)} \to \infty \tag{2.3}$$

If case (ii) holds we say that f is of the same order as g and write it as

$$f \sim g \qquad \text{or} \tag{2.4}$$

$$f = O\,[g] \tag{2.5}$$

If case (i) holds we say that f is of higher order (tends to zero faster) than g and write

$$f < g \qquad \text{or} \tag{2.6}$$

$$f = o\,[g] \tag{2.7}$$

The set of gauge functions $g_n(\varepsilon)$ is ordered such that

$$g_1(\varepsilon) > g_2(\varepsilon) > g_3(\varepsilon) \quad \ldots \tag{2.8}$$

One realizes that there are infinitely many sets of gauge functions possible. The choice of an appropriate set is quite a bit a matter of experience and judgement. Before we go further we

introduce two additional concepts those of asymptotic sequence and series.

Asymptotic Sequence and Series

An asymptotic sequence is one in which the terms are arranged in an ascending order. Thus a sequence $\{\phi_n(\varepsilon)\}$ is asymptotic if $\phi_{n+1} < \phi_n$, that is

$$\lim_{\varepsilon \to 0} \frac{\phi_{n+1}(\varepsilon)}{\phi_n(\varepsilon)} = 0 \qquad (2.9)$$

A series $\sum a_n \phi_n(\varepsilon)$ is termed an asymptotic series if $\{\phi_n\}$ is an asymptotic sequence.

An asymptotic series $\sum a_n \phi_n$ is an asymptotic expansion of a function $f(\bar{x},\varepsilon)$ to N terms if and only if

$$f(\bar{x},\varepsilon) = \sum_1^N a_n(\bar{x})\, \phi_n(\varepsilon) + o\,[\phi_N] \qquad (2.10)$$

that is the remainder after N terms tends to zero faster than ϕ_N as $\varepsilon \to 0$. This can also be put as

$$f(\bar{x},\varepsilon) = \sum_1^{N-1} a_n(\bar{x})\, \phi_n(\varepsilon) + O\,[\phi_N] \qquad (2.11)$$

Here a_m's are given by

$$a_m = \lim_{\varepsilon \to 0} \frac{f(\bar{x},\varepsilon) - \sum_1^{m-1} a_n \phi_n}{\phi_m} \qquad (2.12)$$

Therefore

$$a_1 = \lim_{\varepsilon \to 0} \frac{f}{\phi_1} \qquad (2.13)$$

$$a_2 = \lim_{\varepsilon \to 0} \frac{f - a_1 \phi_1}{\phi_2} \tag{2.14}$$

The Perturbation Technique

The perturbation technique consists of recognizing the parameter ε for whose limiting value the solution is known or can easily be obtained. The actual solution $f(\bar{x},\varepsilon)$ can then be thought of as a perturbation on this basic (or first order) solution. The dependent variables are expanded as an asymptotic series whose first term is the basic solution and higher terms can be written as $f_n(\bar{x})\ \phi_n(\varepsilon)$, where $\{\phi_n(\varepsilon)\}$ is an asymptotic sequence.

The next important step in the technique is to non-dimensionalize and normalize each variable by its characteristic value for the problem. This ensures that all the variables and derivatives in the equation are of order unity, and the relative orders of the various terms are determined by the gauge function which appear as the coefficients of the variables. By equating terms of similar order we obtain a sequence of equations for functions f_n, that can be solved one by one. It is a property of this technique that only the basic equation can be non-linear, all the rest are necessarily linear. Once we solve for f_n we can write the asymptotic expansion $f_n(\bar{x})\ \phi_n(\varepsilon)$. The radius of convergence of this series then determines the domain of validity of this perturbation expansion. As happens sometimes, this domain of validity includes the whole range of $\bar{x}$ we are interested in, and then the problem is termed one of uniform perturbation. On the other hand, if the expansion obtained is not uniformly valid, we term the problem

one in singular perturbation. The reason why the expansion breaks down in certain region of interest lies in the fact that the characteristic quantities used for normalization are not proper for that region and consequently the derivatives and variables are not all of order one. This happens, for example, in case of boundary layer over a flat plate of length L, in which though the characteristic length for the 'potential flow' region is L, that for the 'boundary-layer' region is δ , the boundary-layer thickness. This upsets the seperation of the equation into the sequence of equations for functions f_n. To overcome this difficulty we use different normalizing variables for this region and obtain an expansion valid for it. The procedure for the search of optimum normalizing quantities and the patching or matching of the two expansions at the interface of their regions of validity can be found in any book on singular perturbation such as Cole (5).

In our present study, because of uniform mechanisms of diffusion throughout the flow field, we have a uniform perturbation problem. More on this later.

In the next chapter we pose the problem of a two-dimensional turbulent jet and develop the basic equation and boundary conditions for the problem.

CHAPTER 3

THE TWO-DIMENSIONAL TURBULENT JET

THE TWO-DIMENSIONAL TURBULENT JET

Introduction

A free jet occurs whenever a fluid is discharged from a nozzle or an orifice. For an inviscid flow a jet has one or more surfaces of discontinuities across which such parameters as flow velocities, temperature and mass concentration experience discontinuities. The presence of diffusion effects of viscosity, thermal conduction and mass transfer tends to dissipate the discontinuities so that a sort of boundary layer is formed. The instability associated with the presence of points of inflexion in the velocity profile create eddies which move in a disorderly fashion and give rise to turbulence which further strengthens the momentum, heat and mass diffusion. It is experimentally determined that the jets are almost always turbulent even at quite moderate Reynolds numbers and even when a very smooth flow is maintained initially. We shall be concerned here in this chapter with constructing a model for a steady two-dimensional jet issuing out of a slit into stationary fluid (same fluid). We will also assume isothermal conditions so that thermal transfer and mass transfer effects do not enter the problem.

Figure 1 shows simplified structure of a 2-D jet. The jet issuing out of the slit (gap 'd') spreads as it moves in the x-direction entraining more and more fluid. The flow has some turbulence almost from the very beginning, though in the initial region there appears a potential core, in which velocity is substantially constant. This

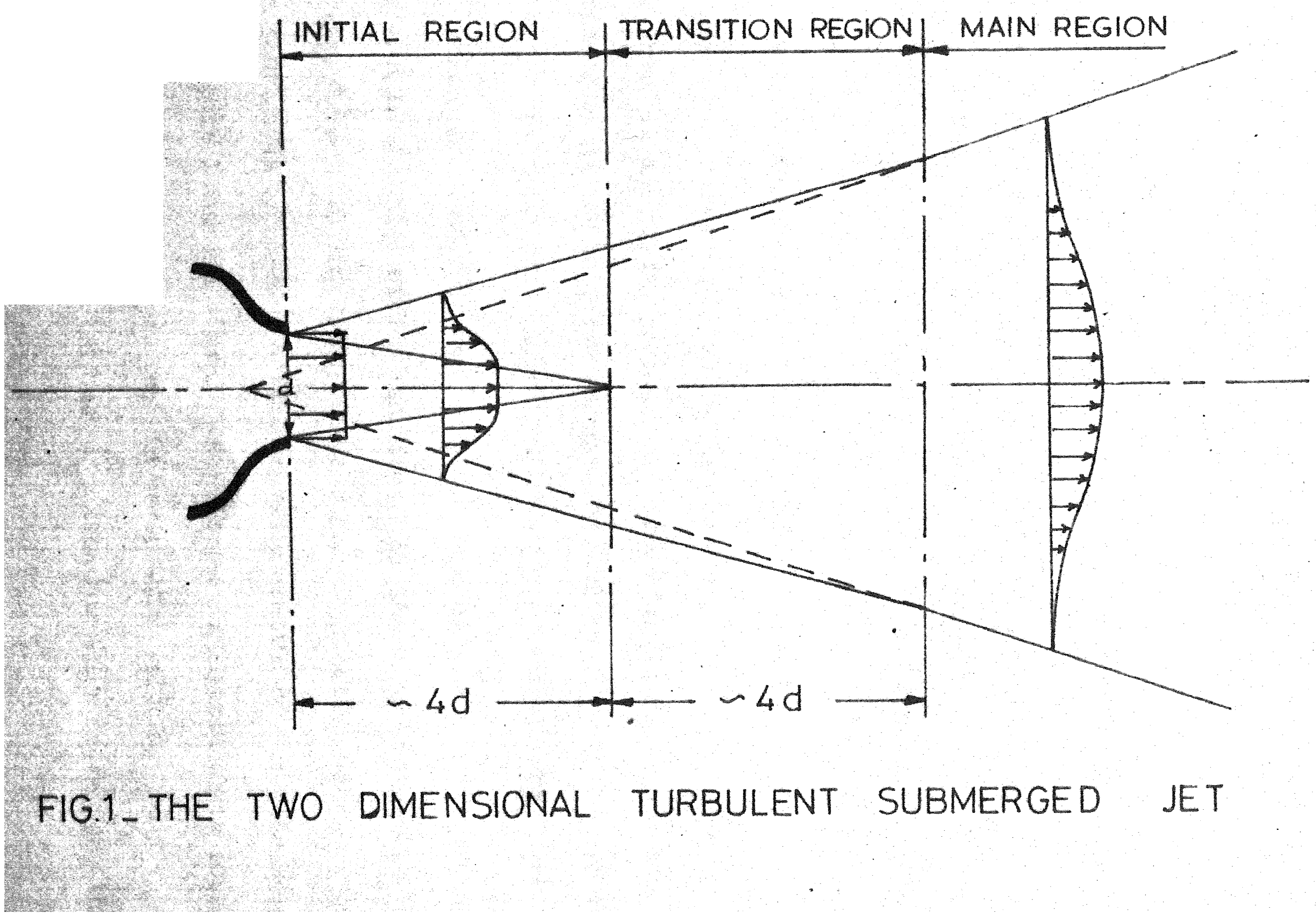

FIG.1_THE TWO DIMENSIONAL TURBULENT SUBMERGED JET

potential core is eaten up by the viscous action and the turbulence as the flow proceeds downstream. This region is known in literature as the mixing region. Next comes the adjustment region beyond $x \approx 4d$ in which the velocity profile adjusts and becomes self-similar in the main region. In this main region which starts at around $x \approx 8d$, the jet becomes similar in appearance to flow from a line source of infinitely small thickness. The velocity profiles are self-similar to atleast the first order. This self-similarity arises from the fact that far away from the slit there appears to be no characteristic length dimension in the problem, except for the eddy size, which should not be relevant at the macroscopic scale.

As the surrounding pressure impresses itself upon the jet stream from all sides, the pressure gradient is very small, and atleast to first order it can be taken as zero, as will be shown. The total momentum in the x-direction therefore remains constant to the first order.

Forthmann studied the velocity profiles of an air jet emerging from a slot 0.03 m high with a velocity of 35 m/sec at distances $x = 0$, 0.2, 0.35, 0.5, 0.625 and 0.75. The results are shown in Fig. 2. They show a continuous broadening of the velocity profile of the jet. The profiles become lower and wider. When plotted in dimensionless form, that is u/u_o vs y/y_c where u_o is centre-line velocity and y_c distance where velocity is half of u_o, then for the main region they fall, more or less on the same curve. See Fig. 3.

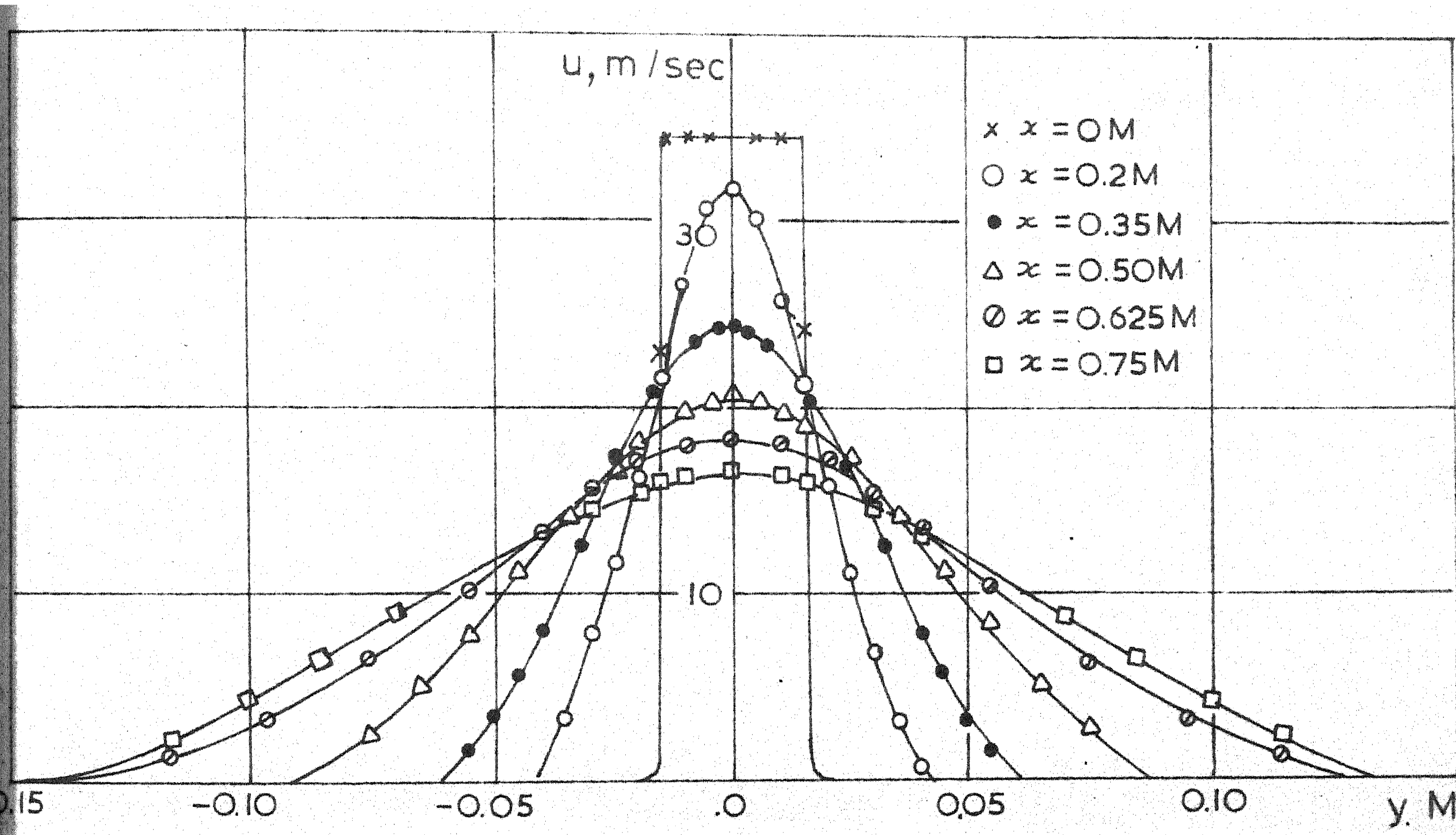

FIG. 2 _ VELOCITY PROFILES AT DIFFERENT SECTIONS . OF A PLANE JET.

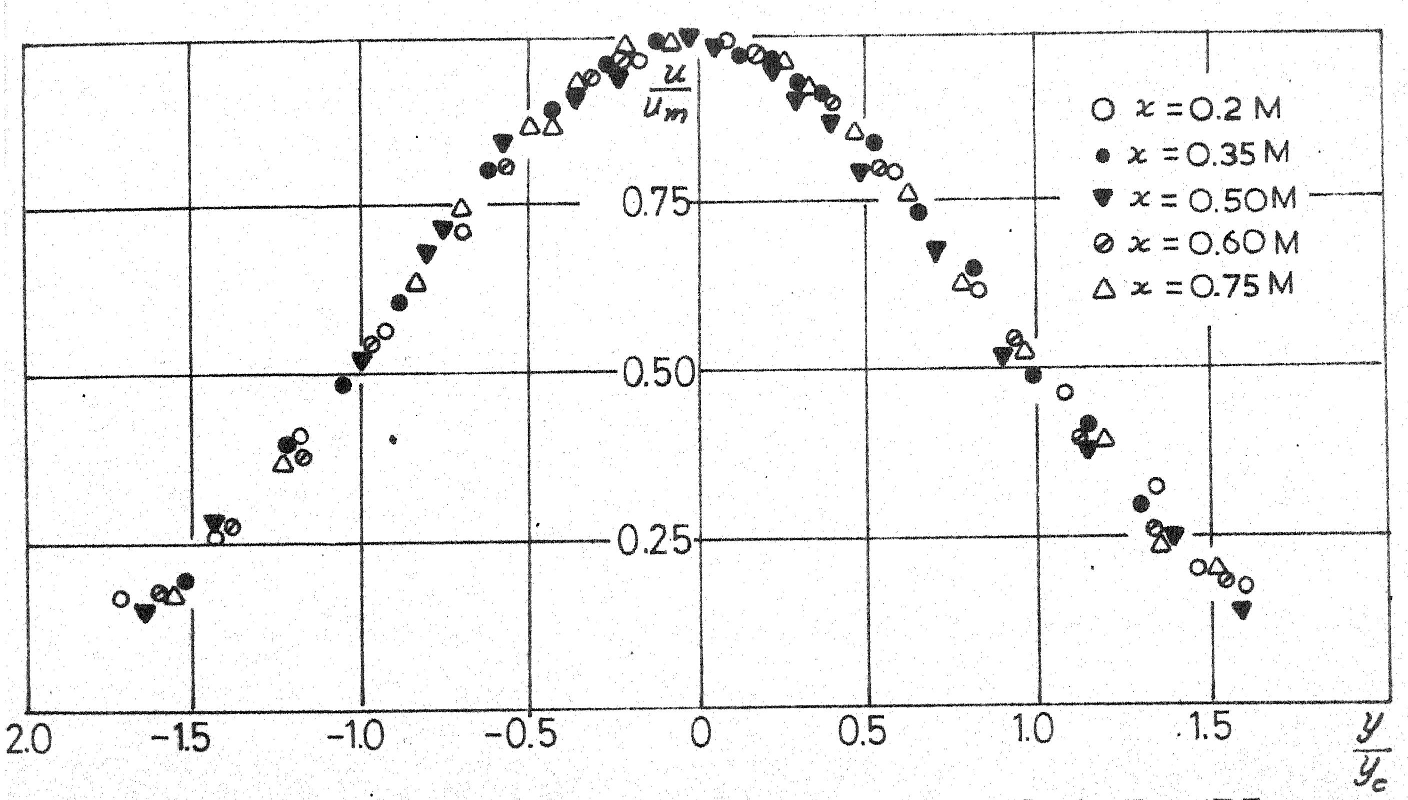

FIG. 3 _ DIMENSIONLESS VELOCITY PROFILE IN PLANE JET

$$u = \bar{u} + u' \qquad \text{and}$$

$$v = \bar{v} + v' \tag{3.1}$$

Prandtl (14) assumed that the rate of spread of jet (measured by the material rate of increase of jet half-width b with time) is directly proportional to the transverse component of fluctuating velocity. Thus

$$\frac{db}{dt} \sim v' \tag{3.2}$$

The similarity of the profiles across the jet also implies that the non-dimensional mixing lengths at the corresponding points should not vary with the axial location. The similarity of velocity profiles is obtained by scaling the transverse length by b, the jet half-width and the velocity by u_o, the centre line velocity (or the maximum velocity). The similarity therefore gives

$$\frac{u}{u_o} = f\left(\frac{y}{b}\right), \quad \text{and,} \tag{3.3}$$

As will be shown in the next chapter, the fluctuation of velocity u' at a given length is approximated by

$$u' = \ell \frac{\partial u}{\partial y} \tag{3.5}$$

and the fluctuations in the transverse component are proportional to (but with opposite sign than) the streamwise component, so that

$$v' \sim -\ell \frac{\partial u}{\partial y} \tag{3.6}$$

Because of similarity of velocity profiles

$$\frac{\partial u}{\partial y} \sim \frac{u_o}{b} \quad , \qquad \text{so that} \tag{3.7}$$

$$\frac{db}{dt} \sim v' \sim -\ell \frac{\partial u}{\partial y} \sim \frac{\ell\, u_o}{b} \sim u_o \tag{3.8}$$

But
$$\frac{db}{dt} = \frac{db}{dx} \cdot \frac{dx}{dt} \sim \frac{db}{dx} u_o \tag{3.9}$$

Thus (3.8) and (3.9) give

$$\frac{db}{dx} \sim 1 \tag{3.10}$$

or
$$b = c\ x \tag{3.11}$$

where c is a constant of proportionality.

The above result with the fact of similarity of velocity profile gives that the constant velocity lines are straight rays issuing out of the point at $x = 0$. If we take the velocity profiles in the fully developed region and extend the rays backwards, they converge to a point slightly upstream of the slit. See Fig. 4.

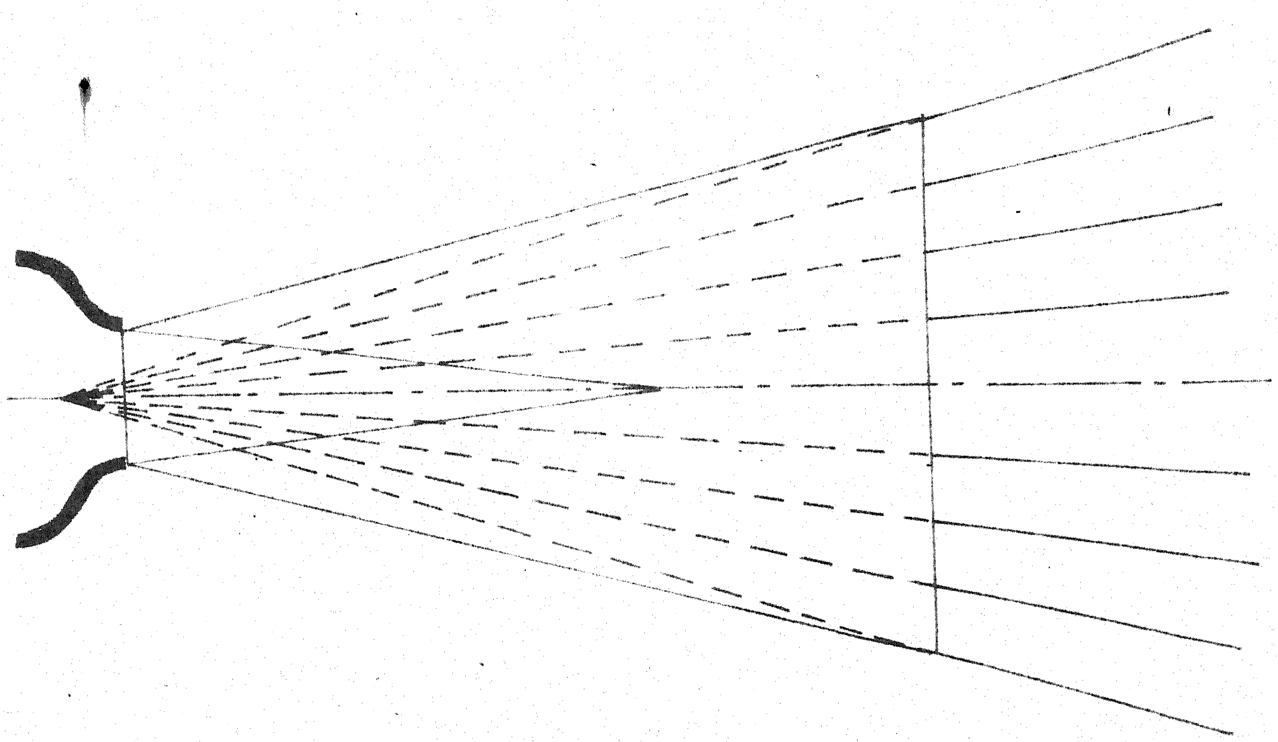

FIG. 4 _EQUAL VELOCITY LINES (u/u_m = CONST.) IN MAIN REGION OF SUBMERGED JET.

Velocity variation along the axis of a submerged jet :

Pressure gradient in a jet is zero atleast to the first order, hence the total momentum of fluid mass per second must be same in all cross sections of the jet. Therefore

$$\int_{Area} \rho u^2 \, dA = Const. \tag{3.12}$$

For a plane submerged jet we can write this as

$$\rho u_o^2 x \int_0^{b/x} \left(\frac{u}{u_o}\right)^2 d\left(\frac{y}{x}\right) = Const. \tag{3.13}$$

Because of similarity of velocity profiles we have

$$\frac{u}{u_o} = f(y/x) \tag{3.14}$$

Hence $$\rho \int_0^{b/x} \left(\frac{u}{u_o}\right)^2 d\left(\frac{y}{x}\right) = Const. \tag{3.15}$$

(3.13) and (3.15) give

$u_o^2 x = Const.,$ and therefore

$$u_o = \frac{Const.}{\sqrt{x}} \tag{3.16}$$

Governing Equation and boundary conditions :

Vorticity transport equation for mean-steady, incompressible, two-dimensional turbulent flow can be obtained from momentum equation as

$$U(\nabla^2\psi),_x + V(\nabla^2\psi),_y = \frac{1}{R}\nabla^4\psi + \tau_{xy},_{yy} + \tau_{xx},_{xy} - \tau_{xy},_{xx} - \tau_{yy},_{xy} \quad \ldots \tag{3.17}$$

where,

$$U = u/U_e \,, \tag{3.18a}$$

$$V = v/U_e \tag{3.18b}$$

$$x = x'/d, \tag{3.19a}$$

$$y = y'/d \tag{3.19b}$$

The Reynolds number 'R' is defined as

$$R = \frac{\rho \, U_e \, d}{\mu} \tag{3.20}$$

and

U_e = Jet exhaust velocity

d = Width of jet slit

x' and y' are physical coordinates, x' being along the jet axis and y' normal to it. u and v are mean velocities along these axes. From the definition of stream function which integrates the continuity equation,

$$U = \frac{\partial \psi}{\partial y} \tag{3.21}$$

$$V = -\frac{\partial \psi}{\partial x} \tag{3.22}$$

τ's are the Reynolds stress components defined as

$$\tau_{xx} = -\overline{u'^2} \tag{3.23a}$$

$$\tau_{xy} = -\overline{u'v'} \tag{3.23b}$$

$$\tau_{yy} = -\overline{v'^2} \qquad \text{where} \tag{3.23c}$$

u' and v' are non-dimensional, fluctuating components of the velocities along the x and y directions. The bar denotes time mean.
In the vorticity equation, commadenotes differentiation.

For a plane-jet the relevant boundary conditions are :

At $y = 0$

$$U = U_o \tag{3.24a}$$

$$V = 0 \tag{3.24b}$$

$$\frac{dU}{dy} = 0 \tag{3.24c}$$

As $y \to \infty$

$$U \to 0 \tag{3.25a}$$

$$u' \to 0 \tag{3.25b}$$

$$v' \to 0 \tag{3.25c}$$

$$\frac{dU}{dy} \to 0 \tag{3.25d}$$

In the next chapter we will set up a perturbation scheme to solve equation (3.17) with boundary conditions (3.24) and (3.25) and obtain solutions upto second order.

CHAPTER 4

PERTURBATION EXPANSIONS OF THE TWO-DIMENSIONAL JET EQUATIONS

PERTURBATION EXPANSIONS OF THE TWO-DIMENSIONAL JET EQUATION

The Perturbation Expansion :

The governing equation with boundary conditions was obtained in the last chapter as :

$$U(\nabla^2\psi),_x + V(\nabla^2\psi),_y = \frac{1}{R}\nabla^4\psi + \tau_{xy},_{yy} + \tau_{xx},_{xy} - \tau_{xy},_{xx} - \tau_{yy},_{xy} \tag{4.1}$$

where,

$$U = \frac{\partial\psi}{\partial y} \quad , \text{ and} \tag{4.2a}$$

$$V = -\frac{\partial\psi}{\partial x} \tag{4.2b}$$

The relevant boundary conditions are:

At $y = 0$:

$$U = U_o \tag{4.3a}$$

$$V = 0 \tag{4.3b}$$

$$\frac{dU}{dy} = 0 \tag{4.3c}$$

As $y \rightarrow \infty$

$$U \rightarrow 0 \tag{4.4a}$$

$$u' \rightarrow 0 \tag{4.4b}$$

$$v' \rightarrow 0 \tag{4.4c}$$

$$\frac{dU}{dy} \rightarrow 0 \tag{4.4d}$$

In free turbulence the distribution of turbulent energy and dissipation are nearly uniform in the main body of the flow, and unlike wall turbulence, are not dominated by gradient diffusion. A large scale convection takes place which transfers turbulence in bulk (25).

In the case of wall shear flow where the vorticity and the turbulence energy are generated at the wall and are then dissipated into the main flow by the viscous and turbulent action, there are two distinct regions of flow, one near the wall in which viscous action predominates, and the other far away from the wall where turbulence dissipation governs the flow. In such a situation we can not construct a uniformly valid perturbation scheme and the problem is necessarily one of singular perturbations. In this case of free shear flow, however, since the viscous and turbulent dissipation are nearly uniformly distributed, we may expect to construct a perturbation model which will apply throughout the flow region. Thus the problem appears to be a regular perturbation problem. We will infact see that a regular perturbation scheme works, and in the following we develop such a scheme.

In this problem we scale the length x such that the new independent length variables X and Y become of the same order. We will use Reynolds number as the parameter, and expand the solutions about $\frac{1}{R} = 0$.

$$X = \Delta(R) \cdot x \tag{4.5}$$

$\Delta(R)$ is a small parameter

$$Y = y \tag{4.6}$$

The expansions of dependent variables can be written as

$$\psi(x,y;R) = E_1(R)\ \psi_1(X,Y) + E_2(R)\ \psi_2(X,Y) + o\ [E_2(R)] \tag{4.7}$$

$$U(x,y;R) = E_1(R)\ \psi_{1,Y}(X,Y) + E_2(R)\ \psi_{2,Y}(X,Y) + o\ [E_2(R)] \tag{4.8}$$

where $E_1(R)$ and $E_2(R)$ are gauge functions belonging to an asymptotic sequence $\{E_n(R)\}$ such that

$$\lim_{R \to \infty} \frac{E_{n+1}(R)}{E_n(R)} \to 0$$

We next expand the Reynolds stresses in asymptotic series. For this purpose, we adopt the closure hypothesis proposed by Prandtl, which is known as Prandtl's new theory of free turbulence. For two-dimensional flow, he approximates the Reynolds stress components by

$$\tau_{xx} = 2\ell^2 \frac{dU}{dy} \cdot \frac{dU}{dx} = 2K \frac{dU}{dx} \qquad (4.9)$$

$$\tau_{xy} \doteq \ell^2 \frac{dU}{dy} \cdot \frac{dU}{dy} = K \frac{dU}{dy} \qquad (4.10)$$

$$\tau_{yy} = 2\ell^2 \frac{dU}{dy} \cdot \frac{dV}{dy} = 2K \frac{dV}{dy} \qquad (4.11)$$

In these expressions by Prandtl's new theory the eddy viscosity coefficient $K = \ell^2 \frac{dU}{dy}$, can be considered independent of the y-coordinate, that is at any station x, it remains constant across the entire cross section of the flow. Prandtl obtained an expression for the eddy viscosity coefficient K by considering the similarity of non-dimensional velocity profiles at various cross sections of the flow which permits us to write

$$\frac{d(U/U_o)}{d(y/b)} = \text{Invariant} \quad \text{for a free jet} \qquad (4.12a)$$

Also the non-dimensional mixing length (expressed in fractions of jet width) is same for corresponding points at different cross sections of flow

$$\text{Therefore } \frac{\ell}{b} = \text{Invariant} \tag{4.12b}$$

Using (4.12a) and (4.12b) we can write for the eddy viscosity coefficient K for a free submerged jet :

$$K = \ell^2 \frac{dU}{dy} = b\, U_o \left(\frac{\ell}{b}\right)^2 \frac{d(U/U_o)}{d(y/b)} \quad \text{or}$$

$$K = \chi_1\, b\, U_o \qquad \text{where} \tag{4.12}$$

χ_1 is a constant. It has been experimentally determined that K remains approximately constant near the centre of the flow field but decreases somewhat towards the edges. Thus the results obtained with this hypothesis match the experimental results near the centre but show some discrepancy towards the edges.

In chapter III we have shown that the centre line velocity decreases as $1/\sqrt{x}$ and the jet half-width increases as x. Thus

$$\begin{aligned} K &= \chi_1\, U_o\, b \\ &= G_1\, C \sqrt{X} \end{aligned} \tag{4.13}$$

where G_1 is an appropriate gauge function undetermined so far and C is a constant.

Using the expansions (4.7) and (4.8) and definitions (4.5) and (4.6) of independent variables, we obtain the various terms of the governing equation (4.1) as

$$U(\nabla^2\psi),x + V(\nabla^2\psi),y = E_1^2\,\nabla(\psi_{1,Y}\,\psi_{1,XYY} - \psi_{1,X}\,\psi_{1,YYY})$$

$$+ E_1\,E_2\,\Delta(\psi_{1,Y}\,\psi_{2,XYY} + \psi_{2,Y}\,\psi_{1,XYY} - \psi_{1,X}\,\psi_{2,YYY}$$

$$- \psi_{2,X}\,\psi_{1,YYY}) + o\,[E_1\,E_2\,\Delta] \qquad (4.14)$$

$$\frac{1}{R}\,(\nabla^4\psi) = \frac{E_1}{R}\,(\psi_{1,YYYY}) + o\,[\frac{E_1}{R}] \qquad (4.15)$$

$$\tau_{xy} = G_1\,C\,\sqrt{X}\,(E_1\,\psi_{1,YY} + E_2\,\psi_{2,YY}) + o\,[G_1\,E_2] \qquad (4.16)$$

$$\tau_{xx} = E_1 G_1\,\Delta(2C\,\sqrt{X}\,\psi_{1,XY}) + o\,[E_1 G_1 \Delta] \qquad (4.17)$$

$$\tau_{yy} = E_1 G_1 \Delta(2C\,\sqrt{X}\,\psi_{1,XY}) + o\,[E_1 G_1 \Delta] \qquad (4.18)$$

On differentiation of (4.16), (4.17) and (4.18) we get

$$\tau_{xy,yy} = E_1 G_1(C\,\sqrt{X}\,\psi_{1,YYYY}) + G_1 E_2(C\,\sqrt{X}\,\psi_{2,YYYY}) + o\,[G_1 E_2] \quad (4.19)$$

$$\tau_{xx,xy} = E_1 G_1\,\Delta^2\,\{\frac{d^2}{dXdY}\,(2C\,\sqrt{X}\,\psi_{1,XY})\} + o\,[E_1 G_1\,\Delta^2] \qquad (4.20)$$

$$\tau_{yy,xy} = E_1 G_1\,\Delta^2\,\{\frac{d^2}{dXdY}\,(2C\,\sqrt{X}\,\psi_{1,XX})\} + o\,[E_1 G_1\,\Delta^2] \qquad (4.21)$$

$$\tau_{xy,xx} = E_1 G_1\,\Delta^2\,\{\frac{d^2}{dX^2}\,(C\,\sqrt{X}\,\psi_{1,YY})\} + o\,[E_1 G_1\,\Delta^2] \qquad (4.22)$$

In free turbulent flows the turbulent stresses are much larger than viscous stresses in the entire region of flow, that is, the viscous stress terms are of a higher order than the Reynolds stress terms (20). The only important contribution from the Reynolds stress terms for the lowest order equation comes from the lowest order term in $\tau_{xy,yy}$, the rest being of a higher order, that is, being much smaller in magnitude.

The lowest order equation must therefore contain the inertia terms and the Reynolds stress terms $\tau_{xy,yy}$. Thus the orders of the lowest order terms in the expansions of the two must be same, and we obtain

$$E_1^2\Delta \sim E_1 \, G_1$$

We note that $E_1 \sim 1$ to satisfy the condition on the jet axis, that is, $U \to U_o$ as $Y \to 0$. Without loss of generality, we can set

$$E_1 = 1 \text{ , and thus obtain} \tag{4.23}$$

$G_1 \sim \Delta$. We can again, without loss of generality, set

$$G_1 = \Delta \tag{4.24}$$

The lowest order equation thus reads :

$$\psi_{1,Y} \; \psi_{1,XYY} - \psi_{1,X} \; \psi_{1,YYY} = C \sqrt{X} \; \psi_{1,YYYY} \tag{4.25}$$

We see that even to second order the only contribution from Reynolds stress terms comes from $\tau_{xy,yy}$, since the order of the second term in expansion of $\tau_{xy,yy}$ (4.19) is lower than the lowest order terms in the other Reynolds stresses. We take the viscous stress terms to be of second order. The second order equation must therefore contain the second order inertia terms (4.14), the second term in the expansion of $\tau_{xy,yy}$ (4.19) and the lowest order term in the expansion of viscous stresses (4.15). Thus the orders of these terms must be the same and we obtain

$$E_1 \, E_2\Delta \sim \frac{E_1}{R} \sim E_2 \, G_1 \tag{4.26a}$$

For the sequence of gauge functions $E_1, E_2 \ldots$ we will choose a power series in the small parameter Δ which gives, $E_2 \sim \Delta$. This choice can be justified by the argument that if the Reynolds stresses, other than $\tau_{xy,yy}$ are to contribute even to the third order equation they must be of the same order as the third order terms in the expansion of inertia forces. For this we must have $E_2 \sim \Delta$. Without loss of generality we can put,

$$E_2 = \Delta \tag{4.26b}$$

We obtain a relation between R and Δ from (4.24) and (4.26). a and b

$$\Delta \sim \frac{1}{\sqrt{R}}, \text{ and without loss of generality}$$

$$\Delta = \frac{1}{\sqrt{R}} \tag{4.27}$$

Hence the first and second order equations from (4.13), (4.14), (4.15) and (4.19) turn out to be

$$\psi_{1,Y}\ \psi_{1,XYY} - \psi_{1,X}\ \psi_{1,YYY} = C\sqrt{X}\ \psi_{1,YYYY} \tag{4.28}$$

$$\psi_{1,Y}\ \psi_{2,XYY} + \psi_{2,Y}\ \psi_{1,XYY} - \psi_{1,X}\ \psi_{2,YYY} - \psi_{2,X}\ \psi_{1,YYY}$$

$$= \psi_{1,YYYY} + C\sqrt{X}\ \psi_{2,YYYY} \tag{4.29}$$

The corresponding boundary conditions are :

$$\text{At } Y = 0 \qquad \begin{array}{ll} \psi_{1,Y} = U_o & \text{(i)} \\ \psi_{1,X} = 0 & \text{(ii)} \\ \psi_{1,YY} = 0 & \text{(iii)} \end{array} \tag{4.28a}$$

As $Y \to \infty$ $\quad \psi_{1,Y} \to 0 \quad$ (i)

$\psi_{1,YY} \to 0 \quad$ (ii) $\quad$ (4.28b)

and for conditions on ψ_2 we have

At $Y = 0$ $\quad \psi_{2,X} = 0 \quad$ (i)

$\psi_{2,YY} = 0 \quad$ (ii) $\quad$ (4.29a)

As $Y \to \infty$ $\quad \psi_{2,Y} \to 0 \quad$ (i)

$\psi_{2,YY} \to 0 \quad$ (ii) $\quad$ (4.29b)

Solution of First Order Equation :

The equation we obtain on integrating the first order equation (4.28) once with respect to Y is of the nature of boundary layer equation with zero pressure gradient and reads as

$$\psi_{1,Y}\,\psi_{1,XY} - \psi_{1,X}\,\psi_{1,YY} = C\sqrt{X}\,\psi_{1,YYY} \tag{4.30}$$

It should be noted here that since the vorticity equation is obtained by differentiating the momentum equation where in the pressure gradient terms disappear, on partial integration of the vorticity transport equation. we should have a function of integration which should depend upon the pressure gradient. Since the pressure gradient along the jet axis is zero to the first order the constant of integration is also zero.

This is the same equation as obtained and solved by Goertler for the two-dimensional turbulent jet based on Prandtl's second hypothesis (8)

with the difference that this is a non-dimensionalized and scaled equation. Adopting the same approach we introduce η and $f_1(\eta)$, such that

$$\psi_1 = A\sqrt{X}\, f_1(\eta) \quad , \qquad \text{where} \tag{4.31a}$$

$$\eta = \frac{BY}{X} \tag{4.31b}$$

A and B are constants.

Using (3.31)a and b we get for the various terms of the first order equation (4.30)

$$\psi_{1,Y} = \frac{AB}{\sqrt{X}}\, f_1'(\eta) \tag{4.32a}$$

$$\psi_{1,YY} = \frac{AB^2}{X\sqrt{X}}\, f_1''(\eta) \tag{4.32b}$$

$$\psi_{1,YYY} = \frac{AB^2}{X^2\sqrt{X}}\, f_1'''(\eta) \tag{4.32c}$$

$$\psi_{1,X} = \frac{A}{2\sqrt{X}}\, f_1(\eta) - \frac{A\eta}{\sqrt{X}}\, f_1'(\eta) \tag{4.32d}$$

$$\psi_{1,XY} = \frac{-AB}{2X\sqrt{X}}\, f_1'(\eta) - \frac{AB\eta}{X\sqrt{X}}\, f_1''(\eta) \tag{4.32e}$$

Substituting (4.32) in (4.30) we obtain

$$-\frac{A^2B^2}{2X^2}\, f_1'^2(\eta) - \frac{A^2B^2}{X^2}\, \eta\, f_1'(\eta)\, f_1''(\eta) - \frac{A^2B^2}{2X^2}\, f_1''(\eta)\, f_1(\eta)$$

$$+ \frac{A^2B^2}{X^2}\, \eta f_1'(\eta)\, f_1''(\eta) = \frac{AB^3C}{X^2}\, f_1'''(\eta)$$

On simplifying and rearranging we get

$$f_1'^2(\eta) + f_1''(\eta)\, f_1(\eta) + \frac{2BC}{A}\, f_1'''(\eta) = 0 \tag{4.33}$$

As we have two free constants A and B we can for convenience put

$$A = 4BC \tag{4.34}$$

with which the first order equation reduces to

$$2f_1'^2(\eta) + 2f_1''(\eta)\, f_1(\eta) + f_1'''(\eta) = 0 \tag{4.35}$$

This equation must be solved subject to the boundary conditions (4.28) a and b which reduce to the following

$$\eta = 0 : \qquad f_1' = 1 \tag{4.36a}$$

$$f_1 = 0 \tag{4.36b}$$

$$\eta \to \infty \qquad f_1' \to 0 \tag{4.37}$$

Integrating (4.35) once we obtain

$$2f_1(\eta)\, f_1'(\eta) + f_1''(\eta) = \text{Const.} = C_1$$

Integrating this once more we obtain

$f_1^2(\eta) + f_1'(\eta) = C_1\eta + C_2$. Using boundary conditions (4.36) and (4.37) we obtain

$$C_1 = 0$$

$$C_2 = 1$$

Therefore, $$f_1^2(\eta) + f_1'(\eta) = 1 \tag{4.38}$$

This equation is same as the one obtained for two-dimensional laminar jets and solution to this is

$$f_1(\eta) = \tanh \eta \tag{4.39a}$$

and
$$\psi_1 = A\sqrt{X}\ \tanh \eta \tag{4.39b}$$

Hence we get for the velocities

$$U_1 = \psi_{1,Y} = \frac{AB}{\sqrt{X}}\ (1 - \tanh^2 \eta) \tag{4.40}$$

$$V_1 = -\ \psi_{1,X} = -\frac{\Delta A}{2\sqrt{X}}\ [\tanh \eta\ - 2\eta\,(1 - \tanh^2 \eta)] \tag{4.41}$$

Thus we see that the non-dimensional velocity profile is self-similar to the first order atleast.

For evaluation of the constants we use the condition that pressure gradient is zero to the first order and therefore the first order momentum flux in x direction must remain constant.

$$\text{Momentum flux} \quad = J = \rho \int u^2\,dy$$

$$\text{Therefore} \quad J \simeq \rho\ d\ U_e^2 \int_{-\infty}^{+\infty} (\psi_{1,Y})^2\ dy \text{ to the first order}$$

$$= \rho\ d\ U_e^2 \int_{-\infty}^{+\infty} \frac{A^2B^2}{X}\ (1 - \tanh^2 \eta)^2\ dy$$

$$= \rho\ d\ U_e^2\ A^2B \int_{-\infty}^{+\infty} (1 - \tanh^2 \eta)^2\ d\eta$$

Let $\xi = \tanh \eta$

Therefore $d\xi = (1 - \tanh^2 \eta)\ d\eta$

Hence $J = \rho\ d\ U_e^2\ A^2B \int_{-1}^{+1} (1 - \xi^2)\ d\xi$

$$= \rho\ d\ U_e^2\ A^2B(4/3)$$

Therefore $\frac{4}{3} A^2B = \frac{J}{\rho\, d\, U_e^2} = 1$, and

$$A = \frac{\sqrt{3}}{2} \cdot \sqrt{1/B} \tag{4.42}$$

From (4.34) and (4.42)

$$C = \frac{A}{4B} = \frac{\sqrt{3}}{8B\sqrt{B}} \tag{4.43}$$

Now the only unknown constant is B and this is evaluated experimentally.

To investigate for the contribution of pressure gradient term to the second order equation let us consider the y-momentum equation

$$U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} = - \frac{\partial p}{\partial y} + \frac{1}{R} \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} \tag{4.44}$$

If we use the expansions of U., V etc. in this equation the lowest order equation becomes

$$- \Delta^2 \left(\psi_{1,Y}\, \psi_{1,XX} - \psi_{1,X}\, \psi_{1,XY} - 2C\sqrt{X}\; \psi_{1,XYY}\right) = - \frac{\partial p}{\partial Y} \tag{4.45}$$

Thus $\frac{\partial p}{\partial Y} \sim \Delta^2$

Since Y is of order one we can write

$$p \sim \Delta^2$$

We next find

$$\frac{\partial p}{\partial \eta} = \frac{\partial p}{\partial Y} \cdot \frac{\partial Y}{\partial \eta}$$

$$= \frac{X}{B} \frac{\partial p}{\partial Y}$$

p is a function of η and x therefore

$$\frac{\partial p}{\partial x} = \Delta \frac{\partial p}{\partial X}$$

$$= \Delta \left(\frac{\partial p}{\partial \eta} \cdot \frac{\partial \eta}{\partial X} + \frac{\partial p}{\partial X}\right)$$

$$= \Delta \left(\frac{\partial p}{\partial X} - \frac{\eta}{B} \frac{\partial p}{\partial Y}\right)$$

The terms in the bracket are each of order Δ^2 so that

$$\frac{\partial p}{\partial x} \sim \Delta^3$$

Thus we see that $\frac{\partial p}{\partial x}$ is of a order higher than second order also, or is smaller in magnitude and can be neglected from the second order equation also.

Solution of Second Order Equation :

Integrating the second order equation (4.29) once with respect to Y we obtain

$$\psi_{1,Y}\,\psi_{2,XY} + \psi_{2,Y}\,\psi_{1,XY} - \psi_{1,X}\,\psi_{2,YY} - \psi_{2,X}\,\psi_{1,YY}$$

$$= \psi_{1,YYY} + C\sqrt{X}\;\psi_{2,YYY} \qquad (4.50)$$

Since the pressure gradient term has been shown to be of an order higher than the second order, we can take the function of integration to be zero.

The first order terms are given by

$$\psi_1 = A\sqrt{X}\;\tanh\eta \quad \text{with } \eta = BY/X \qquad \text{(from eqn. 4.39)}$$

$$\psi_{1,Y} = \frac{AB}{\sqrt{X}} (1 - \tanh^2 \eta)$$

$$\psi_{1,YY} = - \frac{2AB^2}{X\sqrt{X}} \tanh \eta \, (1 - \tanh^2 \eta)$$

$$\psi_{1,YYY} = \frac{2AB^3}{X^2\sqrt{X}} \, [3\tanh^2 \eta \, (1 - \tanh^2 \eta) - (1 - \tanh^2 \eta)]$$

$$\psi_{1,X} = \frac{A}{2\sqrt{X}} \, [\tanh \eta - 2\eta\,(1 - \tanh^2 \eta)] \tag{4.51}$$

$$\psi_{1,XY} = \frac{AB}{2X\sqrt{X}} \, [4\eta \tanh \eta \, (1 - \tanh^2 \eta) - (1 - \tanh^2 \eta)]$$

$$= \frac{AB}{2X\sqrt{X}} \, (1 - \tanh^2 \eta)\,(4\eta \tanh \eta - 1)$$

Substituting from (4.51) in (4.50) we obtain

$$\frac{AB}{X} (1 - \tanh^2 \eta)\, \psi_{2,XY} + \frac{AB}{2X\sqrt{X}} (4\eta \tanh \eta - 1)\,(1 - \tanh^2 \eta)\, \psi_{2,Y}$$

$$- \frac{A}{2\sqrt{X}} \, [\tanh \eta - 2\eta\,(1 - \tanh^2 \eta)] \, \psi_{2,YY} + \frac{2AB^2}{X\sqrt{X}} \tanh \eta \,(1-\tanh^2 \eta)\, \psi_{2,X}$$

$$- C \sqrt{X}\, \psi_{2,YYY} = \frac{2AB^3}{X^2\sqrt{X}} (1 - \tanh^2 \eta)\,(3 \tanh^2 \eta - 1) \tag{4.52}$$

Multiplying through out by $\dfrac{X^2\sqrt{X}}{AB(1 - \tanh^2 \eta)}$ we get

$$X^2 \psi_{2,XY} + X(2\eta \tanh \eta - \tfrac{1}{2})\, \psi_{2,Y} - \frac{X^2}{B} \left[\frac{\tanh \eta}{2(1-\tanh^2 \eta)} - \eta\right] \psi_{2,YY}$$

$$+ 2B \tanh \eta \, \psi_{2,X} - \frac{C X^3 \psi_{2,YYY}}{AB(1-\tanh^2 \eta)} = 2B^2(3 \tanh^2 \eta - 1) \tag{4.53}$$

The solution of this partial differential equation would become simpler if we could reduce it to an ordinary differential equation by similarity arguments. To explore for similarity in the second order

equation we let

$$\psi_2 = A X^a f_2(\eta_2), \text{ and} \tag{4.54a}$$

$$\eta_2 = \frac{BY}{X^b} \tag{4.54b}$$

Using (4.54) we get for the various terms of the second order equation (4.53)

$$\psi_{2,Y} = AB\, X^{a-b}\, f_2'(\eta_2) \tag{4.55a}$$

$$\psi_{2,YY} = AB^2\, X^{a-2b}\, f_2''(\eta_2) \tag{4.55b}$$

$$\psi_{2,YYY} = AB^3\, X^{a-3b}\, f_2'''(\eta_2) \tag{4.55c}$$

$$\psi_{2,X} = a\, A\, X^{a-1}\, f_2(\eta_2) - A\, X^{a-1}\, \eta_2\, f_2'(\eta_2) \tag{4.55d}$$

$$\psi_{2,XY} = a\, A\, B\, X^{a-b-1}\, f_2'(\eta_2) - AB\, X^{a-b-1}\, \eta_2\, f_2''(\eta_2) - AB\, X^{a-b-1}\, f_2'(\eta_2) \tag{4.55e}$$

From (4.55) and (4.53) we see that for the coefficients of $\psi_{2,XY}$; $\psi_{2,Y}$; $\psi_{2,YY}$; $\psi_{2,X}$ and $\psi_{2,YYY}$ to be independent of X we must have

$$a - b - 1 = -2 \tag{4.56a}$$

$$a - b = -1 \tag{4.56b}$$

$$a - 2b = -2 \tag{4.56c}$$

$$a - 1 = -1 \tag{4.56d}$$

$$a - 3b = -3 \tag{4.56e}$$

These conditions are satisfied if we put

$$a = 0 \tag{4.57a}$$

$$b = 1 \tag{4.57b}$$

so that

$$\eta_2 = \eta \tag{4.58}$$

and we obtain

$$\psi_2 = A\, f_2(\eta) \qquad \text{where} \qquad \eta = \frac{BY}{X} \tag{4.59}$$

With these expressions for ψ_2 and η we get

$$\psi_{2,Y} = \frac{AB}{X}\, f_2'(\eta) \tag{4.60a}$$

$$\psi_{2,YY} = \frac{AB^2}{X^2}\, f_2''(\eta) \tag{4.60b}$$

$$\psi_{2,YYY} = \frac{AB^3}{X^3}\, f_2'''(\eta) \tag{4.60c}$$

$$\psi_{2,X} = -\frac{A\eta}{X}\, f_2'(\eta) \tag{4.60d}$$

$$\psi_{2,XY} = -\frac{AB\eta}{X^2}\, f_2''(\eta) - \frac{AB}{X^2}\, f_2'(\eta) \tag{4.60e}$$

Substituting from (4.60) in (4.53) we get the second order equation as

$$-BA\,[\eta f_2''(\eta) + f_2'(\eta)] + A\,B\, f_2'(\eta)\,(2\eta \tanh \eta - \tfrac{1}{2})$$

$$-AB\,\left[\frac{\tanh \eta}{2(1-\tanh^2 \eta)} - \eta\right] f_2''(\eta) - 2AB\eta \tanh \eta\, f_2'(\eta)$$

$$-\left[\frac{B^2 C}{(1-\tanh^2 \eta)}\right] f_2'''(\eta) = 2B^2\,(3 \tanh^2 \eta - 1)$$

or,

$$-\frac{AB\tanh\eta}{2(1-\tanh^2\eta)}\,f_2''(\eta)-\frac{3}{2}AB\,f_2'(\eta)-\frac{B^2C}{(1-\tanh^2\eta)}\,f_2'''(\eta)$$

$$=2B^2(3\tanh^2\eta-1)$$

To simplify this equation, multiply throughout by

$$\frac{(1-\tanh^2\eta)}{B^2C}\;,\qquad\text{or by }\frac{4(1-\tanh^2\eta)}{AB}\quad\text{as}\quad C=\frac{A}{4B}\;.$$

This gives

$$f_2'''(\eta)+2\tanh\eta\; f_2''(\eta)+6(1-\tanh^2\eta)\,f_2'(\eta)$$

$$+\frac{8B}{A}(1-\tanh^2\eta)(3\tanh^2\eta-1)=0 \tag{4.61}$$

We are interested in velocity U_2 which is proportional to f_2'. Thus we can reduce this third order differential equation to one of second order by putting

$$f_2'(\eta)=Z(\eta) \tag{4.62}$$

This gives,

$$Z''(\eta)+2\tanh\eta\,Z'(\eta)+6(1-\tanh^2\eta)\;Z(\eta)$$

$$+\frac{8B}{A}(1-\tanh^2\eta)(3\tanh^2\eta-1)=0 \tag{4.63}$$

At $\eta=0$ we have

$$Z''(0)+6Z(0)-\frac{8B}{A}=0 \tag{4.64}$$

As $\eta\to\infty$ we have

$$Z''(\infty)+2Z'(\infty)=0 \tag{4.65}$$

Let

$$2 \tanh \eta = g_1(\eta) \tag{4.66a}$$

$$6(1 - \tanh^2 \eta) = g_2(\eta) \tag{4.66b}$$

$$\frac{8B}{A} (1 - \tanh^2 \eta)(3 \tanh^2 \eta - 1) = g_3(\eta) \tag{4.66c}$$

Then the second order equation reduces to,

$$Z''(\eta) + g_1(\eta)\, Z'(\eta) + g_2(\eta)\, Z(\eta) + g_3(\eta) = 0 \tag{4.67}$$

The boundary conditions to be satisfied are

At $\eta = 0$: $Z' = 0$ (4.68)

As $\eta \to \infty$: $Z \to 0$ (4.69)

Obtaining an analytical solution to this equation is a difficult task because of the variable coefficients $g_1(\eta)$, $g_2(\eta)$ etc. involved. We will resort to numerical techniques using the principle of superposition, outlined below.

The Numerical Technique

An analytical solution of a second order equation of the type of (4.67) will be of the form

$$Z = C_1 Z_{H1} + C_2 Z_{H2} + Z_P \tag{4.70}$$

where Z_{H1} and Z_{H2} are solutions of the homogeneous part of the differential equation and Z_P is the particular solution. C_1 and C_2 are constants. Because of the linearity of the equation we can solve for Z_{H1}, Z_{H2} and Z_P independently and then add the results, after

determination of the constants C_1 and C_2, to satisfy the boundary conditions.

The homogeneous part of the differential equation (4.67) reads as

$$Z_H''(\eta) + g_1(\eta)\ Z_H'(\eta) + g_2(\eta)\ Z_H(\eta) = 0 \tag{4.71}$$

For the initial values we will organize this as

$$Z_{H1}(0) = 1 \tag{4.72a}$$

$$\left.\frac{dZ_{H1}}{d\eta}\right|_{\eta=0} = 0 \tag{4.72b}$$

and

$$Z_{H2}(0) = 0 \tag{4.73a}$$

$$\left.\frac{dZ_{H2}}{d\eta}\right|_{\eta=0} = 0 \tag{4.73b}$$

For the particular solution the entire equation has to be solved, that is,

$$Z_P''(\eta) + g_1(\eta)\ Z_P'(\eta) + g_2(\eta)\ Z_P(\eta) + g_3(\eta) = 0 \tag{4.74}$$

The initial values for this will be taken as

$$Z_P(0) \quad = 0 \tag{4.75a}$$

$$\left.\frac{dZ_P}{d\eta}\right|_{\eta=0} = 0 \tag{4.75b}$$

The constants C_1 and C_2 in equation (4.70) have to be evaluated from the overall boundary conditions as follows

At $\eta = 0$, if we have $Z = a$ we can write

$$Z(0) = a = C_1 \; Z_{H1}(0) + C_2 \; Z_{H2}(0) + Z_P(0)$$

$$= C_1$$

Therefore $\quad C_1 = a \qquad (4.76)$

As $\eta \to \infty$, if we have $Z = b$ we can write as

$$Z(\infty) = b = C_1 \; Z_{H1}(\infty) + C_2 \; Z_{H2}(\infty) + Z_P(\infty)$$

Therefore $$C_2 = \frac{b - C_1 \; Z_{H1}(\infty) - Z_P(\infty)}{Z_{H2}(\infty)} \qquad (4.77)$$

For our problem the value of Z as $\eta \to \infty$ is zero, that is, $b = 0$. The initial value $Z(0)$ is not known, however from equation (4.64) we see that to start the integration procedure we can take

$$Z(0) \simeq \frac{8B}{6A}$$

It was also found that with the initial values as given by (4.73), the solution of the homogeneous equation was a trivial one, that is, the value remains zero throughout. This caused the solution of C_2 to blow up. To avoid this $Z_{H2}(0)$ was given a small value, of the order of 1×10^{-5}. Z_{H1}, Z_{H2} and Z_P have been solved by the Runge Kutta method. The results obtained have been discussed in the next chapter.

CHAPTER 5

RESULTS AND DISCUSSION

RESULTS AND DISCUSSION

In the last chapters we developed the equations for first and second-order velocity distribution using perturbation theory arguments. One of the main points that emerged in the analysis is that viscosity effects which are negligible compared to the Reynolds stress effects in the first order become significant at the second order. Also it was shown that even to second order the pressure variations are negligible compared to the effects of the other terms in the momentum balance equation. Since for the range $10^2 < Re < 5 \times 10^4$ the second order correction is small, the effect of pressure variation on the jet in this region is really very small. Only for $Re < 10^2$, when the second order correction is substantial and indicates that the third term may need to be considered, can the pressure variation have any possible significance.

The first order velocity profile, as was expected, is independent of the Reynolds number and of x-station. This infact represents the solution in the limit $x \to \infty$ and $Re \to \infty$. For moderately high Reynolds number and for moderately large x/d (as opposed to large x/d and Re) this profile needs correction. We calculated, using the procedure outlined in the previous chapter the second order correction to the basic velocity profile for four x-stations (10, 15, 20 and 25 jet widths from the slit) and five jet-Reynolds numbers (10^2, 10^3, 5×10^3, 10^4 and 5×10^4). The results obtained are presented as the following figures.

Figure 5 shows the second-order corrections across the jet for $Re = 5 \times 10^4$ and $x/d = 25$. It should be noted that the effect of viscosity is to reduce the velocity near the centre line and to increase the velocity at the edges. This is the typical effect expected of viscosity-to diffuse the velocity gradient so as to make the velocity differentials smaller. Figures 6 and 7 show the second order corrections across the jet for $Re = 10^3$ and $x/d = 25$; and $Re = 10^3$ and $x/d = 15$ respectively.

Figure 8 shows the variation of corrected centre-line velocity with Reynolds number at a given x-station ($x/d = 20$). The effect obtained is small, increasing with decreasing Reynolds number. As Reynolds number increases the corrected centre-line velocity tends to its first order value. This is also in line with the explanations since the decreasing Reynolds number mean the increasing viscous effects.

Figure 9 shows the variation in the second order correction to the centre-line velocity with x-station for a typical Reynolds number (10^3). Figure 10 also shows the percentage correction in the normalized velocity (centre-line velocity taken as unity) at a given non-dimensional y-location ($\eta = 1$) and a given Reynolds number = 10^4. Both the figures exhibit a decrease in correction as x increases. It can be inferred that the perturbation results obtained are valid for x/d between 8 and 25. Beyond 25, the unperturbed results are good enough, while below 8, the perturbation scheme breaks down because of the absence of full development.

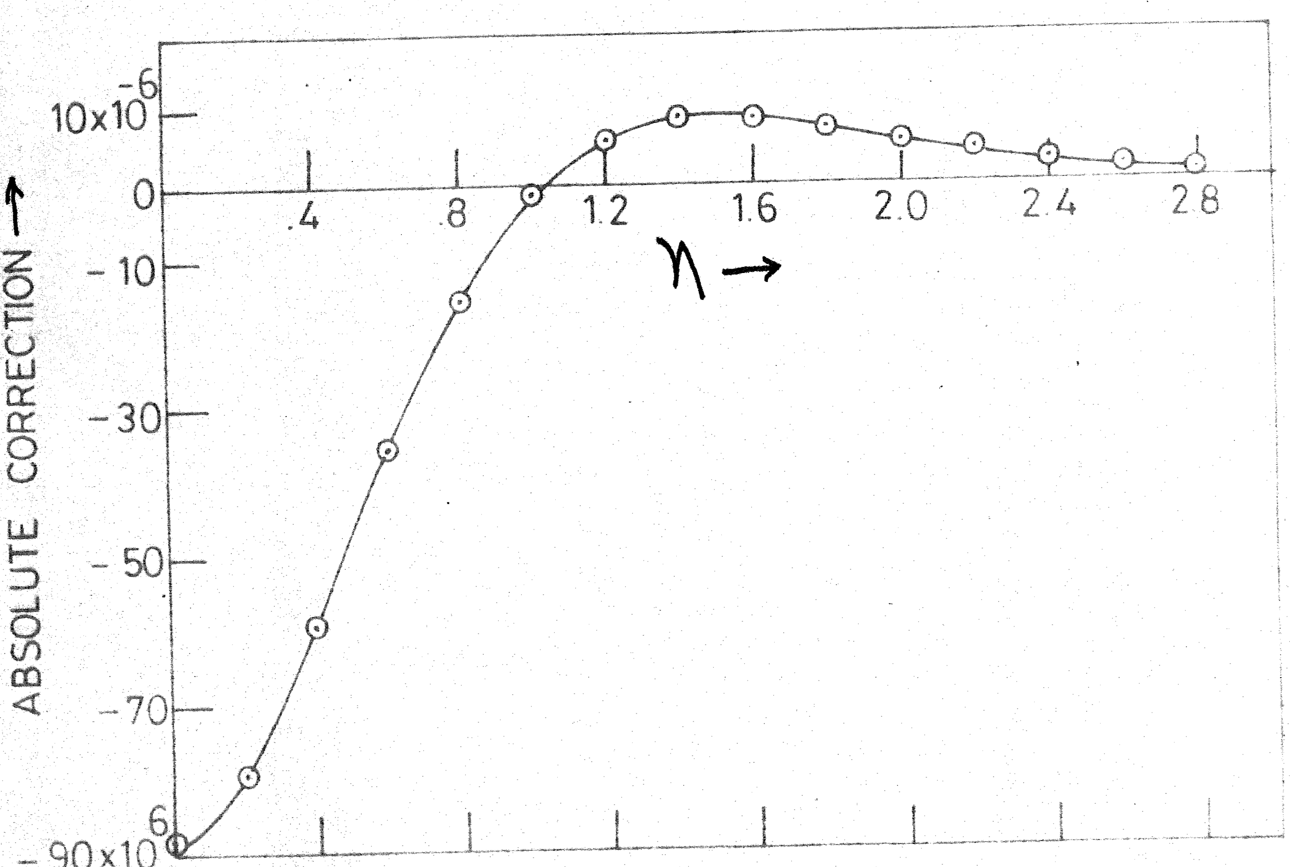

FIG. 5_VARIATION IN CORRECTION DUE TO VISCOSITY WITH η

REYNOLDS NO: 5×10^4 STATION (x/d) : 25

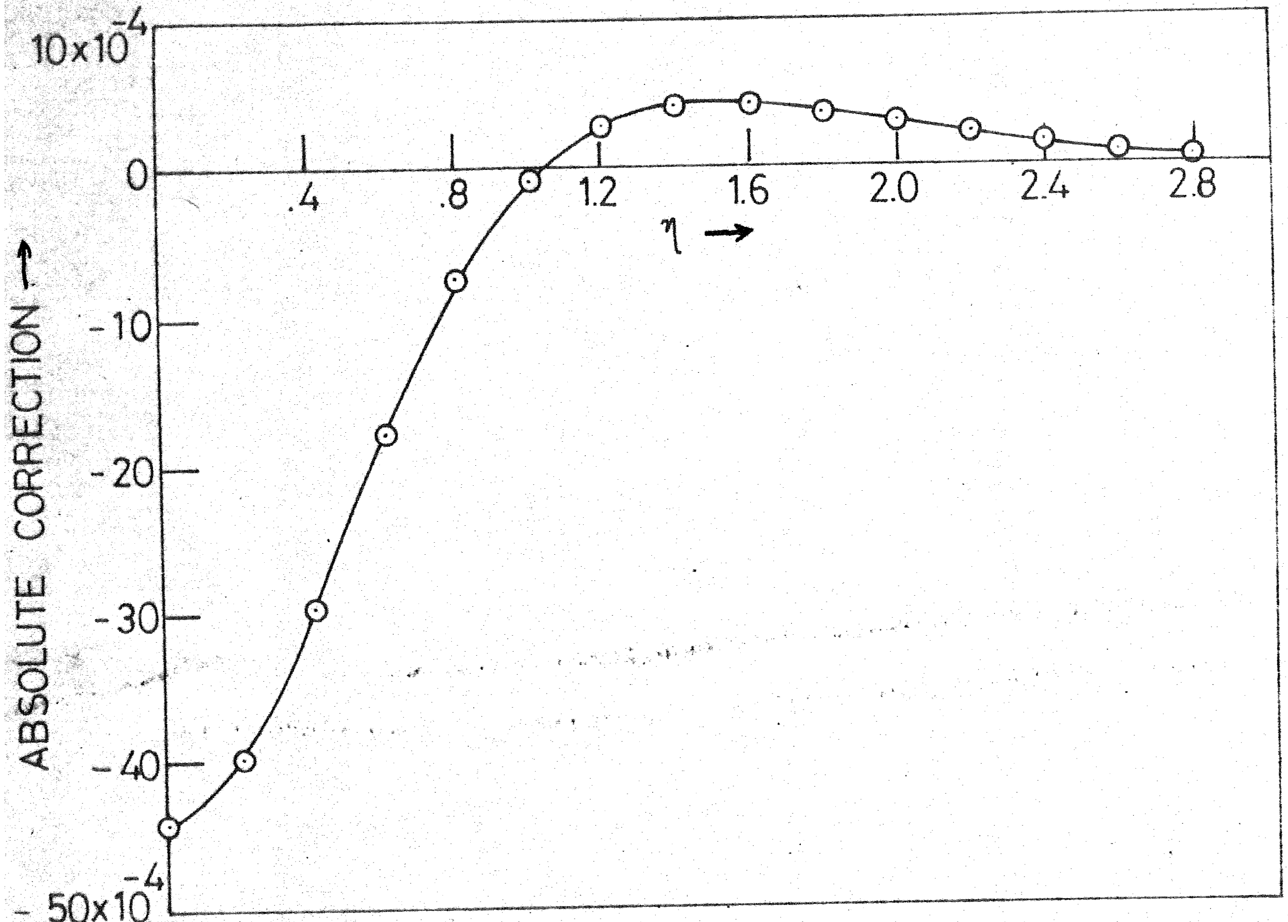

FIG.6_VARIATION IN CORRECTION DUE TO VISCOSITY WITH η
REYNOLDS NUMBER : 10^3 STATION (x/d) : 25

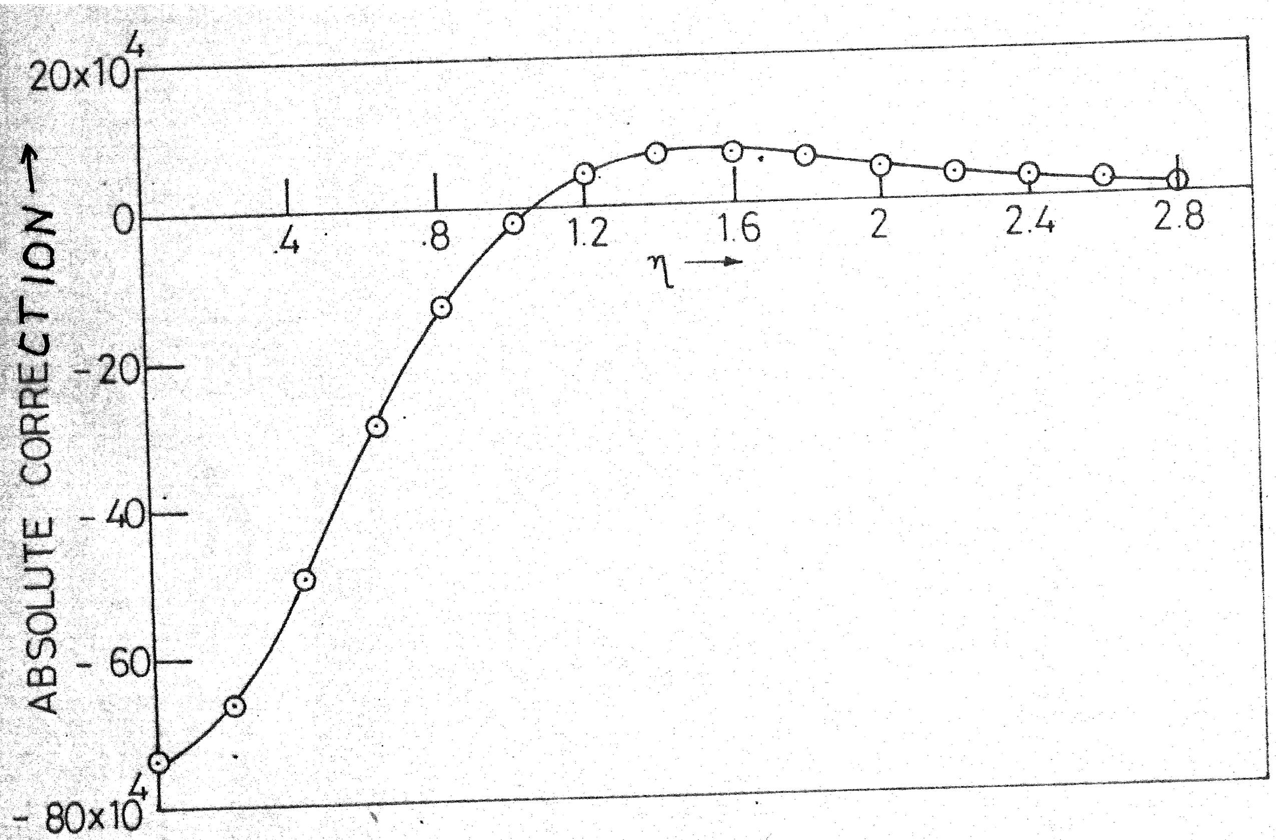

FIG.7 – VARIATION IN CORRECTION DUE TO VISCOSITY WITH η
REYNOLDS NUMBER : 10^3 STATION (x/d) : 15

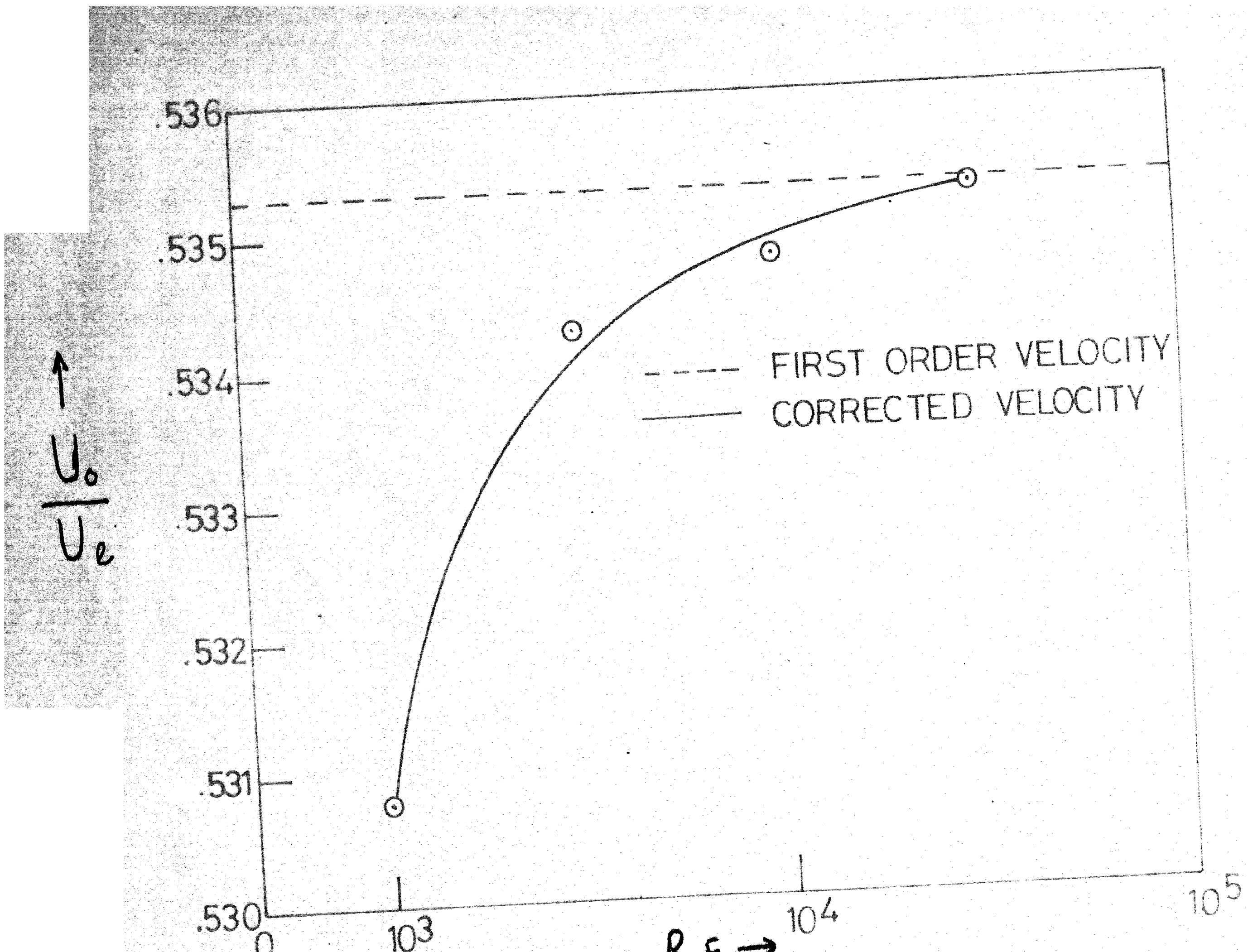

FIG. 8_ VARIATION IN CENTRE LINE VELOCITY WITH REYNOLDS NUMBER ; STATION (x/d) : 20

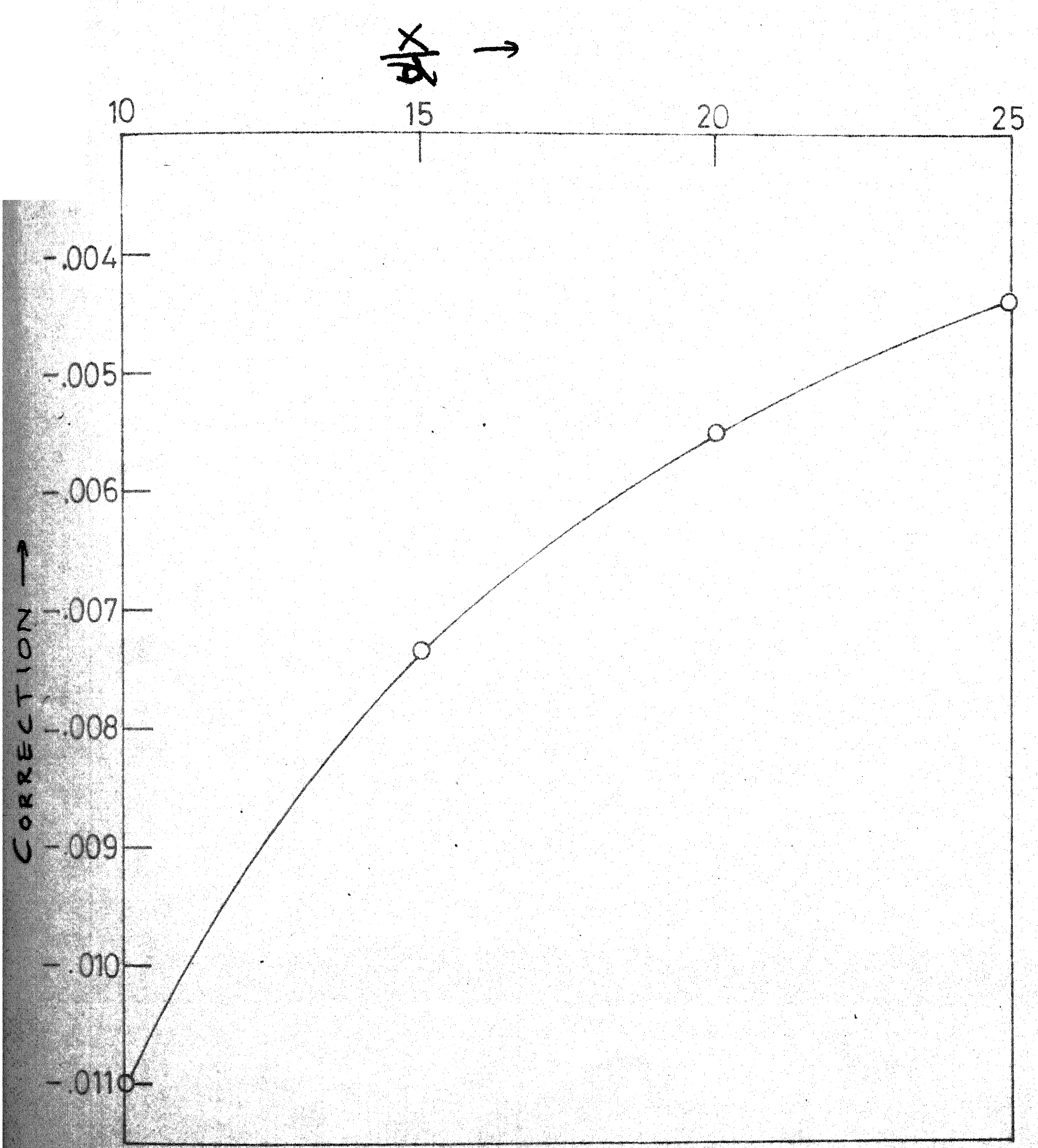

FIG. 9_ VARIATION IN CORRECTION TO CERTRE LINE VELOCITY WITH STATION (x/d).
REYNOLDS NUMBER : 10^3

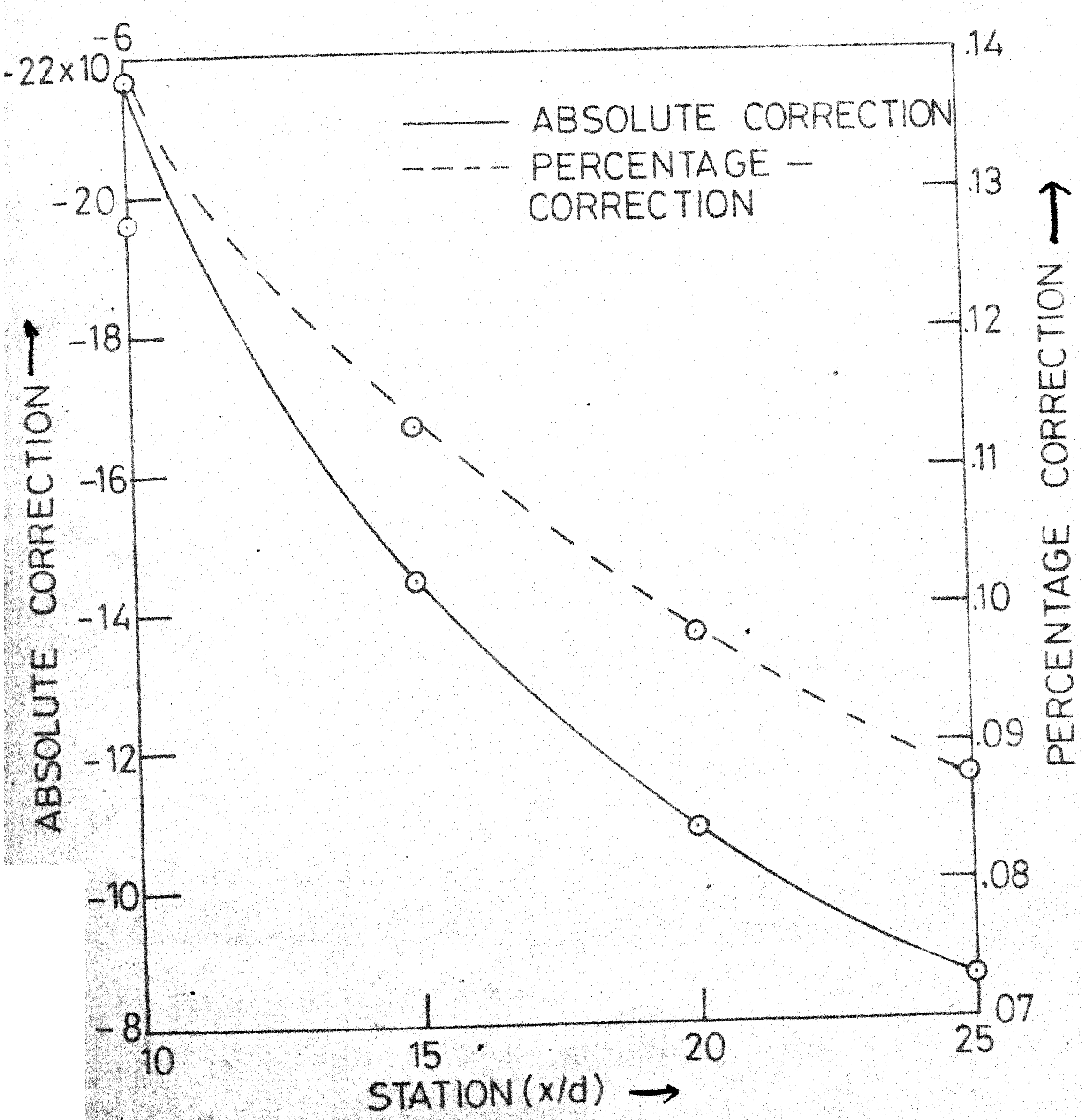

FIG.10 _ VARIATION IN CORRECTION DUE TO VISCOSITY WITH STATION; REYNOLDS NUMBER : 10^4; η : 1

Figure 11 plots the corrections in the velocity profile, normalized with respect to the corrected centre-line velocity. This is plotted for a given x-location ($x/d = 20$) and a given Reynolds number (5×10^3). The significance of this curve is the fact that this brings out the nature of the second order corrections as would be observed in experimental results where the curves are plotted with the velocities normalized with respect to the observed centre-line velocity. Comparing figure 5 with the data reported in literature (20) we note that it does infact represent the experimental results for values of non-dimensional $\eta = y/d$ less that 1.2 ($\eta \sim Y/X$ less than approximately 1.15). For higher values of η the result breaks down. This is due to the difficulty associated with the closure hypothesis. The closure hypothesis due to Prandtl which we use here asserts that $K = \ell \frac{du}{dy}$ is constant across the width of the jet. This is not really true for η greater than approximately 1.15 beyond which the value reduces. Another restriction is imposed by the fact that the one undetermined constant involved in the analysis is dependent on the jet size and is not a universal constant (10).

These results are valid in the Reynolds number range $10^2 < Re < 10^4$ and for x-stations $8 < x/d < 25$. For $x/d > 25$ and $Re > 10^4$ the first order velocity profile (based on infinitely large Reynolds number and full development assumptions) can be taken as valid. For $x/d < 8$ and $Re < 10^2$, the jet model breaks down and the results obtained are no longer valid.

The dots and circles in the graphs do not indicate experimental values.

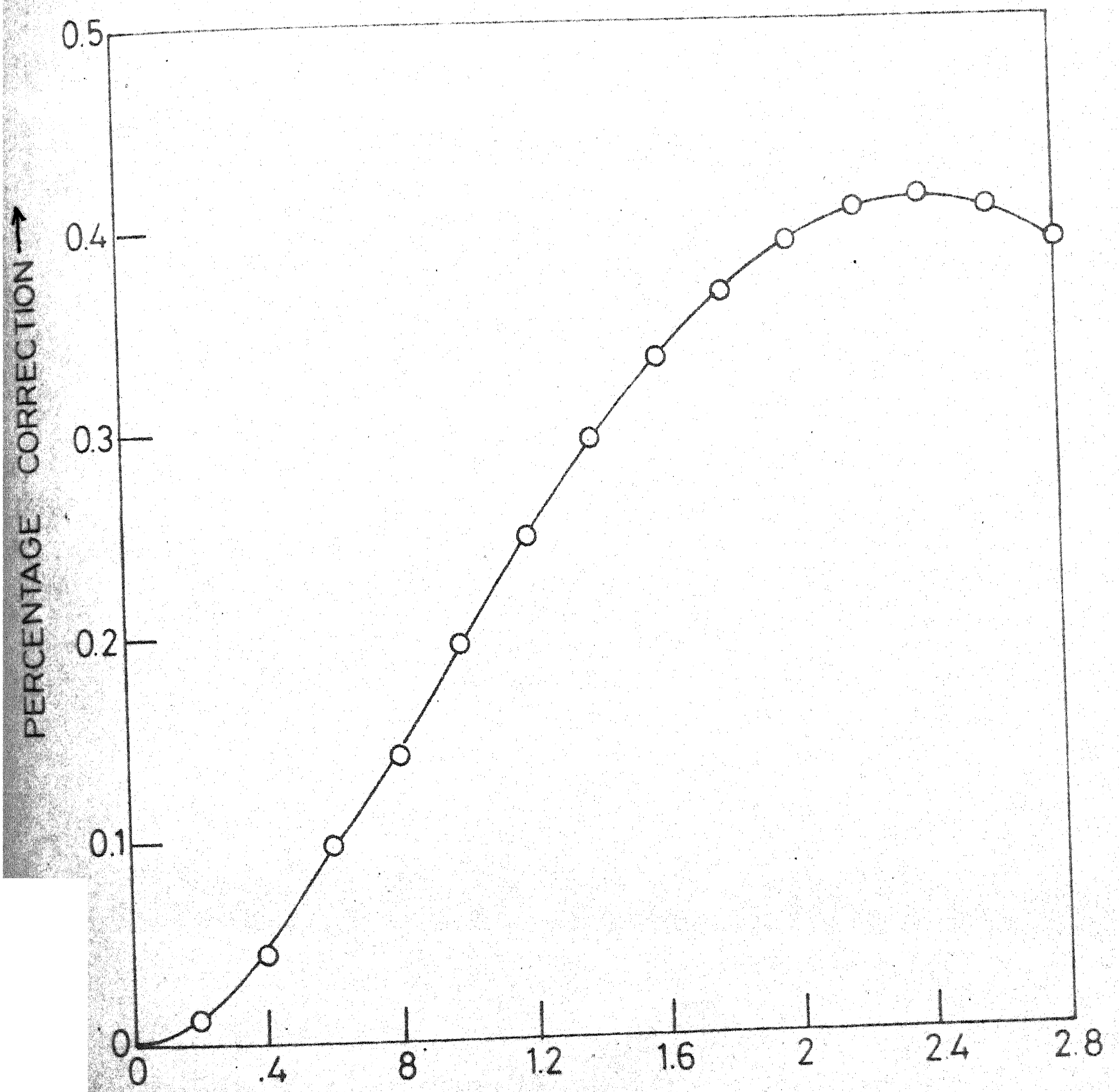

FIG. II_ VARIATION IN NORMALIZED PERCENTAGE CORRECTION WITH η; REYNOLDS NUMBER: 5×10^3 STATION (x/d): 20

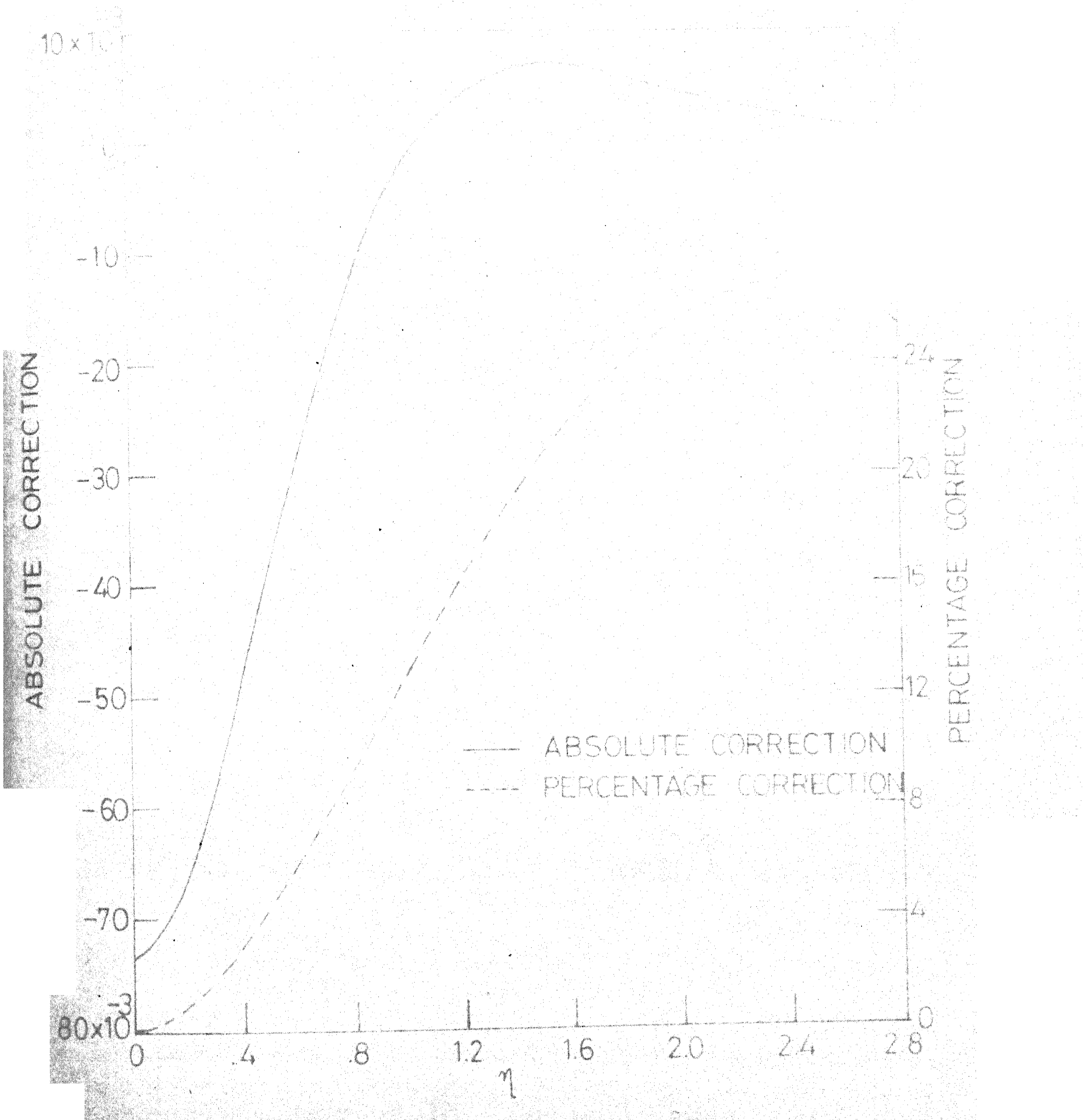

FIG 12 - VARIATION IN CORRECTION DUE TO VISCOSITY WITH η ; REYNOLDS NUMBER $=10^2$
STATION $(x/d) = 15$

ABSTRACT

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      SOLUTION OF DIFFERENTIAL EQUATION FOR HIGHER ORDER EFFECTS IN TURB JETS
      X IS THE SIMILARITY PARAMETER ETA IN OUR ANALYSIS
      DIMENSION X(250),ZH1(250),YH1(250),ZH2(250),YH2(250)
      DIMENSION U2(250),U1NL(250),UTNL(250)
      DIMENSION ZP(250),YP(250),E2U2(250)
      DIMENSION U1(250),CRTN(250)
      G1(X) = 2.-TANH(X)
      G2(X) = 6.*(1.-TANH(X)**2)
      G3(X) = (8.*B/A)*(1.-TANH(X)**2)*(3.*TANH(X)**2-1.)
      READ5,RE , STDN
    5 FORMAT(E6.0,F5.0)
      DELTA = 1./SQRT(RE)
      SIGMA = 7.67
      B = SIGMA * DELTA
      A = SQRT(3.0)/(2.0*SQRT(B))
      C = SQRT(3.0)/(8.0*B**1.5)
      PRINT60,A,B,C
   60 FORMAT(1X,*A=*,F25.8,*B=*,F25.8,*C=*,F25.8)
      PRINT 55, RE, STDN
   55 FORMAT(5X,*RE=*,E10.1,5X,*STDN=*,F6.1)
      ZH1(1) = 1.
      YH1(1) = 0.
      X(1) = 0.
      STN = STDN*DELTA
      H = 0.05
      DO 80 I = 1,240
      DK1 = H * YH1(I)
      DL1 = -H*(G1(X(I))*YH1(I)+G2(X(I))*ZH1(I))
      DK2 = H*(YH1(I) + DL1/2.)
      DL2 = -H*(G1(X(I)+H/2.)*(YH1(I)+DL1/2.)+G2(X(I)+H/2.)*(ZH1(I)+DK1/
     12.))
      DK3 = H*(YH1(I) + DL2/2.)
      DL3 = -H*(G1(X(I)+H/2.)*(YH1(I) + DL2/2.) + G2(X(I) + H/2.)*(ZH1(I
     1) + DK2/2.))
      DK4 = H*(YH1(I) + DL3)
      DL4 = -H*(G1(X(I) + H)*(YH1(I) + DL3) +  G2(X(I) + H)*(ZH1(I) + DK
     13))
      DK = (DK1 + 2.*DK2 + 2.*DK3 + DK4)/6.0
      DL = (DL1 + 2.*DL2 + 2.*DL3 + DL4)/6.0
      ZH1(I + 1) = ZH1(I) + DK
      YH1(I + 1) = YH1(I) + DL
   80 X(I + 1) = X(I) + H
      ZH2(1) = 0.00001
      YH2(1) = 0.
      DO 90 I = 1,240
      DK1 = H*YH2(I)
      DL1 = -H*(G1(X(I))*YH2(I) + G2(X(I))*ZH2(I))
      DK2 = H*(YH2(I) + DL1/2.)
     1(I) + DK2/2.))
      DK4 = H*(YH2(I) + DL3)
      DL3=-H*(G1(X(I)+H/2.)*(YH2(I)+DL2/2.)+G2(X(I)+H/2.)*(ZH2
```

```
      DL4 = -H*(G1(X(I) +H)*(YH2(I) + DL3) + G2(X(I) + H)*(ZH2(I) + DK3)
     1)
      DK = (DK1 + 2.*DK2 + 2.*DK3 + DK4)/6.D
      DL = (DL1 + 2.*DL2 + 2.*DL3 + DL4)/6.D
      ZH2(I + 1) = ZH2(I) + DK
      YH2(I + 1) = YH2(I) + DL
   90 X(I + 1) = X(I) + H
      ZP(1) = 0.
      YP(1) = 0.
      DO 30 I = 1,240
      DK1 = H*YP(I)
      DL1 = -H*(G1(X(I))*YP(I) + G2(X(I))*ZP(I) + G3(X(I)))
      DK2 = H*(YP(I) + DL1/2.)
      DL2 = -H*(G1(X(I)+H/2.)*(YP(I)+DL1/2.)+G2(X(I)+H/2.)*(ZP(I)+DK1/2.
     1) + G3(X(I) + H/2.))
      DK3 = H*(YP(I) + DL2/2.)
      DL3 = -H*(G1(X(I)+H/2.)*(YP(I)+DL2/2.) + G2(X(I)+H/2.)*(ZP(I)+DK2/
     12.) + G3(X(I) + H/2.))
      DK4 = H*(YP(I) + DL3)
      DL4 = -H*(G1(X(I)+H)*(YP(I)+DL3)+G2(X(I)+H)*(ZP(I)+DK3)+G3(X(I)+H)
     1)
      DK = (DK1 + 2.*DK2 + 2.*DK3 + DK4)/6.D
      DL = (DL1 + 2.*DL2 + 2.*DL3 + DL4)/6.D
      ZP(I + 1) = ZP(I) + DK
      YP(I + 1) = YP(I) + DL
   30 X(I + 1) = X(I) + H
      DO 40 I = 1,240
      U1(I) = (A*B*(1.-TANH(X(I))**2))/SQRT(STN)
   40 X(I + 1) = X(I) + H
      C1 = (8.*B)/(6.*A)
      C2 = -(ZP(240)   + C1*ZH1(240))/ZH2(240)
      DO 95 I = 1,60,4
      E2U2(I) = A*B*DELTA*(C1*ZH1(I) + C2*ZH2(I) + ZP(I) )/STN
      U1NL(I) = U1(I) / U1(1)
      UTNL(I) = (U1(I) + E2U2(I))/(U1(1) + E2U2(1))
      CRTN(I) = (100.*(UTNL(I) - U1NL(I)))/U1NL(I)
   95 PRINT 18, X(I) ,U1(I),E2U2(I),U1NL(I),UTNL(I),CRTN(I)
   18 FORMAT(2X,*ETA=*,F5.2,2X,*U1=*,F12.3,2X,*E2U2=*,F12.3,2X,*U1NL=*,F
     112.3,2X,*UTNL=*,F12.3,2X,*CRTN=*,F12.3)
      GO TO *
   50 STOP
      END
ENTRY
```

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